Solution to Practice 1m

B6(a) Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix} \right\}$ be another basis for R^3. To find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates, we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to the standard basis $\mathcal{S}$. But we immediately see that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$$

and so we have

$$Q = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & -1 & -3 \end{bmatrix}$$

To find the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates, we use Theorem 4.4.2, which tells us that $P = Q^{-1}$, and then we use the matrix inverse algorithm to find Q^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 0 & -4 & -1 & 1 & 0 \\ 0 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2 \updownarrow R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 0 & 1 \\ 0 & 0 & -4 & -1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ (-1)R_2 \\ (-1/4)R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1/4 & -1/4 & 0 \end{array}\right] \begin{array}{c} R_1 - 5R_3 \\ R_2 - 3R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/4 & 5/4 & 0 \\ 0 & 1 & 0 & -3/4 & 3/4 & -1 \\ 0 & 0 & 1 & 1/4 & -1/4 & 0 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/4 & 2/4 & 1 \\ 0 & 1 & 0 & -3/4 & 3/4 & -1 \\ 0 & 0 & 1 & 1/4 & -1/4 & 0 \end{array}\right]$$

And so we see that $P = Q^{-1} = \begin{bmatrix} 2/4 & 2/4 & 1 \\ -3/4 & 3/4 & -1 \\ 1/4 & -1/4 & 0 \end{bmatrix}$.

B6(b) Let $\mathcal{S} = \{1, x, x^2\}$ be the standard basis for P_2, and let $\mathcal{B} = \{-1 + 2x^2, 1 + x + x^2, 1 - x - 3x^2\}$ be another basis for P_2. To find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates, we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to the standard basis $\mathcal{E}$. But we immediately see that

$$[-1+2x^2]_{\mathcal{S}} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad [1+x+x^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad [1-x-3x^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

and so we have

$$Q = \left[\; [-1+2x^2]_{\mathcal{S}} \quad [1+x+x^2]_{\mathcal{S}} \quad [1-x-3x^2]_{\mathcal{S}} \;\right] = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & -3 \end{bmatrix}$$

To find the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates, we use Theorem 4.4.2, which tells us that $P = Q^{-1}$, and then we use the matrix inverse algorithm to find Q^{-1}:

$$\left[\begin{array}{rrr|rrr} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \begin{matrix} (-1)R_1 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 2 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ R_3 - 3R_2 \end{matrix} \sim \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right] \begin{matrix} \\ \\ (1/2)R_3 \end{matrix}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right] \begin{matrix} R_1 + R_3 \\ R_2 + R_3 \\ \\ \end{matrix} \sim \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 0 & -3/2 & 1/2 \\ 0 & 1 & 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right] \begin{matrix} R_1 + R_2 \\ \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right]$$

And so we see that $P = Q^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1/2 & 1/2 \\ 1 & -3/2 & 1/2 \end{bmatrix}$.

B6(c) Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be the standard basis for the vector space $\mathbb{V}$ of 2×2 diagonal matrices, and let $\mathcal{B} = \left\{ \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \right\}$ be another basis for $\mathbb{V}$. To find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates, we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to the standard basis $\mathcal{S}$. But we immediately see that

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

and so we have

$$Q = \left[\; \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}_{\mathcal{S}} \quad \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}_{\mathcal{S}} \;\right] = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$$

To find the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates, we use Theorem 4.4.2, which tells us that $P = Q^{-1}$, and then we use the matrix inverse algorithm to find Q^{-1}:

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \begin{matrix} R_1 + R_2 \\ \end{matrix} \sim \left[\begin{array}{cc|cc} 1 & 8 & 1 & 1 \\ -2 & 3 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 + 2R_1 \end{matrix}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 8 & 1 & 1 \\ 0 & 19 & 2 & 3 \end{array}\right] \begin{matrix} \\ (1/19)R_2 \end{matrix} \sim \left[\begin{array}{cc|cc} 1 & 8 & 1 & 1 \\ 0 & 1 & 2/19 & 3/19 \end{array}\right] \begin{matrix} R_1 - 8R_2 \\ \end{matrix}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3/19 & -5/19 \\ 0 & 1 & 2/19 & 3/19 \end{array}\right]$$

And so we see that $P = Q^{-1} = \begin{bmatrix} 3/19 & -5/19 \\ 2/19 & 3/19 \end{bmatrix}$.

D2 Suppose $\mathbb{V}$ is a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Then $\mathcal{C} = \{\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_1\}$ is also a basis for $\mathbb{V}$. The matrix P such that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ is

$$\begin{bmatrix} [\mathbf{v}_1]_{\mathcal{C}} & [\mathbf{v}_2]_{\mathcal{C}} & [\mathbf{v}_3]_{\mathcal{C}} & [\mathbf{v}_4]_{\mathcal{C}} \end{bmatrix}$$

To find the columns of P, we note that:

$$\begin{aligned} \mathbf{v}_1 &= 0(\mathbf{v}_3) + 0(\mathbf{v}_2) + 0(\mathbf{v}_4) + 1(\mathbf{v}_1) \\ \mathbf{v}_2 &= 0(\mathbf{v}_3) + 1(\mathbf{v}_2) + 0(\mathbf{v}_4) + 0(\mathbf{v}_1) \\ \mathbf{v}_3 &= 1(\mathbf{v}_3) + 0(\mathbf{v}_2) + 0(\mathbf{v}_4) + 0(\mathbf{v}_1) \\ \mathbf{v}_4 &= 0(\mathbf{v}_3) + 0(\mathbf{v}_2) + 1(\mathbf{v}_4) + 0(\mathbf{v}_1) \end{aligned}$$

This means that

$$[\mathbf{v}_1]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad [\mathbf{v}_2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad [\mathbf{v}_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad [\mathbf{v}_4]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

And so we have that $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.