Solution to Practice 1l

A1(a) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $x_1, x_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 5 \\ -2 \\ 5 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix}$$

The second coefficient tells us that $x_2 = -2$, while the fourth coordinate tells us the $x_1 = 3$. This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. We can verify this by seeing that

$$3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 5 \\ 3 \end{bmatrix}$$

To find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$, we need to find $y_1, y_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} -1 \\ 3 \\ -1 \\ 2 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ y_2 \\ y_1 - y_2 \\ y_1 \end{bmatrix}$$

The second coefficient tells us that $y_2 = 3$, while the fourth coordinate tells us the $y_1 = 2$. This tells us that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. We can verify this by seeing that

$$2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 2 \end{bmatrix}$$

A1(b) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$\begin{aligned} -2 + 8x + 5x^2 &= s_1(1 + x + x^2) + s_2(1 + 3x + 2x^2) + s_3(4 + x^2) \\ &= (s_1 + s_2 + 4s_3) + (s_1 + 3s_2)x + (s_1 + 2s_2 + s_3)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{lllll} s_1 & +s_2 & +4s_3 & = -2 \\ s_1 & +3s_2 & & = 8 \\ s_1 & +2s_2 & +s_3 & = 5 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. We need to find $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{aligned} -4+8x+4x^2 &= t_1(1+x+x^2)+t_2(1+3x+2x^2)+t_3(4+x^2) \\ &= (t_1+t_2+4t_3)+(t_1+3t_2)x+(t_1+2t_2+t_3)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{lllll} t_1 & +t_2 & +4t_3 & = -4 \\ t_1 & +3t_2 & & = 8 \\ t_1 & +2t_2 & +t_3 & = 4 \end{array}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 4 & -2 & -4 \\ 1 & 3 & 0 & 8 & 8 \\ 1 & 2 & 1 & 5 & 4 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c|c} 1 & 1 & 4 & -2 & -4 \\ 0 & 2 & -4 & 10 & 12 \\ 0 & 1 & -3 & 7 & 8 \end{array}\right] (1/2)R_2$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 1 & 4 & -2 & -4 \\ 0 & 1 & -2 & 5 & 6 \\ 0 & 1 & -3 & 7 & 8 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c|c} 1 & 1 & 4 & -2 & -4 \\ 0 & 1 & -2 & 5 & 6 \\ 0 & 0 & -1 & 2 & 2 \end{array}\right] -R_3$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 1 & 4 & -2 & -4 \\ 0 & 1 & -2 & 5 & 6 \\ 0 & 0 & 1 & -2 & -2 \end{array}\right] \begin{array}{l} R_1 - 4R_3 \\ R_2 + 2R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 6 & 4 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & -2 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 5 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & -2 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$. We can verify this by seeing that

$$-2+8x+5x^2 = 5(1+x+x^2)+(1+3x+2x^2)-2(4+x^2)$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$. We can verify this by seeing that

$$-4+8x+4x^2 = 2(1+x+x^2)+2(1+3x+2x^2)-2(4+x^2)$$

A1(c) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = s_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + s_3 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} s_1+2s_3 & s_1+s_2 \\ s_1+s_2 & s_2-s_3 \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the system

$$\begin{array}{cccl} s_1 & & +2s_3 & = 0 \\ s_1 & +s_2 & & = 1 \\ s_1 & +s_2 & & = 1 \\ & s_2 & -s_3 & = 2 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. We need to find $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} -4 & 1 \\ 1 & 4 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} t_1+2t_3 & t_1+t_2 \\ t_1+t_2 & t_2-t_3 \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the system

$$\begin{array}{cccl} t_1 & & +2s_3 & = -4 \\ t_1 & +t_2 & & = 1 \\ t_1 & +t_2 & & = 1 \\ & t_2 & -t_3 & = 4 \end{array}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 2 & 0 & -4 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 & 4 \end{array}\right] \begin{array}{l} \\ R_3 - R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 2 & 0 & -4 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 4 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 \updownarrow R_4 \end{array}$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & -2 & 1 & 5 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_3 - R_2 \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & -2 & 1 & 5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 + 2R_3 \end{array} \sim$$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} -4 & 1 \\ 1 & 4 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

A1(d) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \end{bmatrix} = s_1\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + s_2\begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} s_1 & s_1 + 2s_2 & -s_2 \\ s_2 & s_1 + 3s_s & s_1 - s_2 \end{bmatrix}$$

From the (11) component, we see that $s_1 = 1$, and from the (21) component we see that $s_2 = 1$. This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix}$$

To find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$, we need to find $t_1, t_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 3 & -1 & 2 \\ -2 & -3 & 5 \end{bmatrix} = t_1\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + t_2\begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} t_1 & t_1 + 2t_2 & -t_2 \\ t_2 & t_1 + 3t_s & t_1 - t_2 \end{bmatrix}$$

From the (11) component, we see that $t_1 = 3$, and from the (21) component we see that $t_2 = -2$. This tells us that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} 3 & -1 & 2 \\ -2 & -3 & 5 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix}$$

A1(e) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$\begin{aligned} 2 + x - 5x^2 + x^3 - 6x^4 &= s_1(1 + x^2 + x^4) + s_2(1 + x + 2x^2 + x^3 + x^4) + s_3(x - x^2 + x^3 - 2x^4) \\ &= (s_1 + s_2) + (s_2 + s_3)x + (s_1 + 2s_2 - s_3)x^2 + (s_2 + s_3)x^3 + (s_1 + s_2 - 2s_3)x^4 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} s_1 & +s_2 & & = 2 \\ & s_2 & +s_3 & = 1 \\ s_1 & +2s_2 & -s_3 & = -5 \\ & s_2 & +s_3 & = 1 \\ s_1 & +s_2 & -2s_3 & = -6 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. We need to find $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{aligned} 1 + x + 4x^2 + x^3 + 3x^4 &= t_1(1 + x^2 + x^4) + t_2(1 + x + 2x^2 + x^3 + x^4) + t_3(x - x^2 + x^3 - 2x^4) \\ &= (t_1 + t_2) + (t_2 + t_3)x + (t_1 + 2t_2 - t_3)x^2 + (t_2 + t_3)x^3 + (t_1 + t_2 - 2t_3)x^4 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} t_1 & +t_2 & & = 1 \\ & t_2 & +t_3 & = 1 \\ t_1 & +2t_2 & -t_3 & = 4 \\ & t_2 & +t_3 & = 1 \\ t_1 & +t_2 & -2t_3 & = 3 \end{array}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{rrr|r|r} 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & -6 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_1 \\ R_4 - R_2 \\ R_5 - R_1 \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -7 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -8 & 2 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -8 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -8 & 2 \end{array}\right] (-1/2)R_3 \sim \left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -8 & 2 \end{array}\right] \begin{array}{l} R_2 - R_3 \\ \\ \\ \\ R_5 + 2R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 5 & -1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}$. We can verify this by seeing that

$$2+x-5x^2+x^3-6x^4 = 5(1+x^2+x^4)-3(1+x+2x^2+x^3+x^4)+4(x-x^2+x^3-2x^4)$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$. We can verify this by seeing that

$$1+x+4x^2+x^3+3x^4 = -1(1+x^2+x^4)+2(1+x+2x^2+x^3+x^4)-1(x-x^2+x^3-2x^4)$$

B1(a) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $x_1, x_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 5 \\ 1 \\ -2 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_1 + x_2 \\ 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$$

Setting the components equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrr} x_1 & +2x_2 & = 5 \\ x_1 & +x_2 & = 1 \\ 2x_1 & +x_2 & = -2 \\ x_1 & +x_2 & = 1 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. For this, we need to find $y_1, y_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 + 2y_2 \\ y_1 + y_2 \\ 2y_1 + y_2 \\ y_1 + y_2 \end{bmatrix}$$

Setting the components equal to each other, we see that this is equivalent to the system

$$\begin{aligned} y_1 \quad +2y_2 &= 0 \\ y_1 \quad +y_2 &= 1 \\ 2y_1 \quad +y_2 &= 3 \\ y_1 \quad +y_2 &= 1 \end{aligned}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{cc|c|c} 1 & 2 & 5 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{cc|c|c} 1 & 2 & 5 & 0 \\ 0 & -1 & -4 & 1 \\ 0 & -3 & -12 & 3 \\ 0 & -1 & -4 & 1 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & -3 & -12 & 3 \\ 0 & -1 & -4 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \\ R_4 + R_2 \end{array} \sim \left[\begin{array}{cc|c|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} 5 \\ 1 \\ -2 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

B1(b) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix} = s_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + s_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + s_3 \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} s_1 + s_2 & s_1 + s_3 \\ s_2 + s_3 & -s_1 + s_2 + 2s_3 \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrr} s_1 & +s_2 & & = 1 \\ s_1 & & +s_3 & = -3 \\ & s_2 & +s_3 & = 2 \\ -s_1 & +s_2 & +2s_3 & = 3 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. We need to find $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} -1 & 0 \\ 3 & 7 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 & t_1 + t_3 \\ t_2 + t_3 & -t_1 + t_2 + ts_3 \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrr} t_1 & +t_2 & & = -1 \\ t_1 & & +t_3 & = 0 \\ & t_2 & +t_3 & = 3 \\ -t_1 & +t_2 & +2t_3 & = 7 \end{array}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{rrr|r|r} 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 3 \\ -1 & 1 & 2 & 3 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 + R_1 \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -4 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 4 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \\ R_4 + 2R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r|r} 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -4 & 1 \\ 0 & 0 & 2 & -2 & 4 \\ 0 & 0 & 4 & -4 & 8 \end{array}\right] \begin{array}{l} \\ -R_2 \\ (1/2)R_3 \\ \\ \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 4 & -1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -4 & 8 \end{array}\right] \begin{array}{l} \\ R_2 + R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r|r} 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - 1\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$. We can verify this by seeing that

$$\begin{bmatrix} -1 & 0 \\ 3 & 7 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

B1(c) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$\begin{aligned} -1 + 3x - x^2 &= s_1(x + x^2) + s_2(-x + 3x^2) + s_3(1 + x - x^2) \\ &= (s_3) + (s_1 - s_2 + s_3)x + (s_1 + 3s_2 - s_3)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} & & s_3 & = -1 \\ s_1 & -s_2 & +s_3 & = 3 \\ s_1 & +3s_2 & -s_3 & = -1 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. We need to find $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{aligned} 3 + 2x^2 &= t_1(x + x^2) + t_2(-x + 3x^2) + t_3(1 + x - x^2) \\ &= (t_3) + (t_1 - t_2 + t_3)x + (t_1 + 3t_2 - t_3)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} & & t_3 & = 3 \\ t_1 & -t_2 & +t_3 & = 0 \\ t_1 & +3t_2 & -t_3 & = 2 \end{array}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{rrr|r|r} 0 & 0 & 1 & -1 & 3 \\ 1 & -1 & 1 & 3 & 0 \\ 1 & 3 & -1 & -1 & 2 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_3 \\ \\ \\ \end{array} \left[\begin{array}{rrr|r|r} 1 & 3 & -1 & -1 & 2 \\ 1 & -1 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrr|r|r} 1 & 3 & -1 & -1 & 2 \\ 0 & -4 & 2 & 4 & -2 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ R_2 - 2R_3 \\ \\ \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 3 & 0 & -2 & 5 \\ 0 & -4 & 0 & 6 & -8 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right] \begin{array}{c} \\ (-1/4)R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrr|r|r} 1 & 3 & 0 & -2 & 5 \\ 0 & 1 & 0 & -3/2 & 2 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right] \begin{array}{c} R_1 - 3R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 0 & 0 & 5/2 & -1 \\ 0 & 1 & 0 & -3/2 & 2 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5/2 \\ -3/2 \\ -1 \end{bmatrix}$. We can verify this by seeing that

$$-1 + 3x - x^2 = (5/2)(x + x^2) - (3/2)(-x + 3x^2) - (1 + x - x^2)$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$. We can verify this by seeing that

$$3 + 2x^2 = -(x + x^2) + 2(-x + 3x^2) + 3(1 + x - x^2)$$

B1(d) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$\begin{aligned} 3 - 3x^2 &= s_1(1 + 2x + 2x^2) + s_2(-3x - 3x^2) + s_3(-3 - 3x) \\ &= (s_1 - 3s_3) + (2s_1 - 3s_2 - 3s_3)x + (2s_1 - 3s_2)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrr} s_1 & & -3s_3 & = 3 \\ 2s_1 & -3s_2 & -3s_3 & = 0 \\ 2s_1 & -3s_2 & & = -3 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. We need to find $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{aligned} 1 + x^2 &= t_1(1 + 2x + 2x^2) + t_2(-3x - 3x^2) + t_3(-3 - 3x) \\ &= (t_1 - 3t_3) + (2t_1 - 3t_2 - 3t_3)x + (2t_1 - 3t_2)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{aligned} t_1 \quad & & -3t_3 &= 1 \\ 2t_1 & -3t_2 & -3t_3 &= 0 \\ 2t_1 & -3t_2 & &= 1 \end{aligned}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{rrr|r|r} 1 & 0 & -3 & 3 & 1 \\ 2 & -3 & -3 & 0 & 0 \\ 2 & -3 & 0 & -3 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 0 & -3 & 3 & 1 \\ 0 & -3 & 3 & -6 & -2 \\ 0 & -3 & 6 & -9 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r|r} 1 & 0 & -3 & 3 & 1 \\ 0 & -3 & 3 & -6 & -2 \\ 0 & 0 & 3 & -3 & 1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \\ \end{array} \sim \left[\begin{array}{rrr|r|r} 1 & 0 & 0 & 0 & 2 \\ 0 & -3 & 0 & -3 & -3 \\ 0 & 0 & 3 & -3 & 1 \end{array}\right] \begin{array}{l} \\ (-1/3)R_2 \\ (1/3)R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r|r} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1/3 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. We can verify this by seeing that

$$3 - 3x^2 = 0(1 + 2x + 2x^2) + (-3x - 3x^2) - (-3 - 3x)$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 1/3 \end{bmatrix}$. We can verify this by seeing that

$$1 + x^2 = 2(1 + 2x + 2x^2) + (-3x - 3x^2) + (1/3)(-3 - 3x)$$

B1(e) To find the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$, we need to find $s_1, s_2, s_3 \in \mathbb{R}$ such that

$$\begin{aligned} 2 - 2x + 5x^2 - x^3 - 5x^4 &= s_1(1 + x + x^3) + s_2(1 + 2x^2 + x^3 + x^4) + s_3(x^2 + x^3 + 3x^4) \\ &= (s_1 + s_2) + (s_1)x + (2s_2 + s_3)x^2 + (s_1 + s_2 + s_3)x^3 + (s_2 + 3s_3)x^4 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{lllll} s_1 & +s_2 & & = 2 \\ s_1 & & & = -2 \\ & 2s_2 & +s_3 & = 5 \\ s_1 & +s_2 & +s_3 & = -1 \\ & s_2 & +3s_3 & = -5 \end{array}$$

But before we complete this process, let's look at how we find the coordinate vector of $\mathbf{y}$ with respect to $\mathcal{B}$. We need to find $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$\begin{aligned} -1 - 3x + 3x^2 - 2x^3 - x^4 &= t_1(1 + x + x^3) + t_2(1 + 2x^2 + x^3 + x^4) + t_3(x^2 + x^3 + 3x^4) \\ &= (t_1 + t_2) + (t_1)x + (2t_2 + t_3)x^2 + (t_1 + t_2 + t_3)x^3 + (t_2 + 3t_3)x^4 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{lllll} t_1 & +t_2 & & = -1 \\ t_1 & & & = -3 \\ & 2t_2 & +t_3 & = 3 \\ t_1 & +t_2 & +t_3 & = -2 \\ & t_2 & +3t_3 & = -1 \end{array}$$

Now, since these two systems have the same coefficient matrix, we can solve them simultaneously by row reducing the following double-augmented matrix:

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 2 & -1 \\ 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 1 & 5 & 3 \\ 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -1 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -2 & -3 \\ 1 & 1 & 0 & 2 & -1 \\ 0 & 2 & 1 & 5 & 3 \\ 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & -5 & -1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - R_1 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 2 & 1 & 5 & 3 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -5 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - R_2 \\ R_5 - R_2 \end{array} \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 9 & 1 & -3 & -1 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 3 & -9 & -3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_3 \\ R_5 - 3R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 9 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This tells us that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix}$. We can verify this by seeing that

$$2-2x+5x^2-x^3-5x^4 = -2(1+x+x^3)+4(1+2x^2+x^3+x^4)-3(x^2+x^3+3x^4)$$

We also have that $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$. We can verify this by seeing that

$$-1-3x+3x^2-2x^3-x^4 = -3(1+x+x^3)+2(1+2x^2+x^3+x^4)-(x^2+x^3+3x^4)$$