Lecture 1n

Linear Mappings

(pages 227-228)

Let's return now to a concept we developed in our study of $\mathbb{R}^n$–linear mappings. Because, as with bases, linear mappings will be of much greater use in the general world of vector spaces than they were in simply the $\mathbb{R}^n$ spaces. We will, of course, start with a definition.

Definition: If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over $\mathbb{R}$, a function $L : \mathbb{V} \to \mathbb{W}$ is a **linear mapping** if it satisfies the linearity properies
(L1) $L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$
(L2) $L(t\mathbf{x}) = tL(\mathbf{x})$
for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $t \in \mathbb{R}$. If $\mathbb{W} = \mathbb{V}$, then L may be called a **linear operator**.

Note that these two properties can be combined into one statement: $L(t\mathbf{x}+\mathbf{y}) = tL(\mathbf{x}) + L(\mathbf{y})$.

Example: The mapping $L : M(2,2) \to P_2$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$ is a linear mapping, because

$$\begin{aligned} L(t\mathbf{x} + \mathbf{y}) &= L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & 2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= L\left(\begin{bmatrix} ta_1 + a_2 & tb_1 + b_2 \\ tc_1 + c_2 & td_1 + d_2 \end{bmatrix}\right) \\ &= (ta_1 + a_2 + tb_1 + b_2) + (ta_1 + a_2 + tc_1 + c_2)x + (ta_1 + a_2 + td_1 + d_2)x^2 \\ &= (ta_1 + tb_1) + (ta_1 + tc_1)x + (ta_1 + td_1)x^2 + (a_2 + b_2) + (a_2 + c_2)x + (a_2 + d_2)x^2 \\ &= t((a_1 + b_1) + (a_1 + c_1)x + (a_1 + d_1)x^2) + ((a_2 + b_2) + (a_2 + c_2)x + (a_2 + d_2)x^2) \\ &= tL\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & 2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= tL(\mathbf{x}) + L(\mathbf{y}) \end{aligned}$$

Example: The mapping $M : P_3 \to P_3$ defined by $L(a + bx + cx^2 + dx^3) = a^2 + abx + acx^2 + adx^3$ is not linear. Consider, for example, that

$$L(1+2x+x^2+2x^4) = 1+2x+x^2+2x^4 \quad \text{and} \quad L(4-x+2x^2-x^3) = 16-4x+8x^2-4x^3$$

and so we have that

$$\begin{aligned} L(1 + 2x + x^2 + 2x^4) + L(4 - x + 2x^2 - x^3) &= (1 + 2x + x^2 + 2x^4) + (16 - 4x + 8x^2 - 4x^3) \\ &= 17 - 2x + 19x^2 - 2x \end{aligned}$$

but that

$$\begin{aligned} L((1+2x+x^2+2x^4)+(4-x+2x^2-x^3)) &= L(5+x+3x^2+x^3) \\ &= 25+5x+15x^2+5x^3 \end{aligned}$$

and so we have that $L(1+2x+x^2+2x^4)+L(4-x+2x^2-x^3) \neq L((1+2x+x^2+2x^4)+(4-x+2x^2-x^3))$. And this means that L is not closed under addition, so it is not a linear mapping.

It happens that L is also not closed under scalar multiplication. Consider, for example, that $L(1+2x+x^2+2x^4) = 1+2x+x^2+2x^4$, so $5L(1+2x+x^2+2x^4) = 5(1+2x+x^2+2x^4) = 5+10x+5x^2+10x^3$, but $L(5(1+2x+x^2+2x^4)) = L(5+10x+5x^2+10x^4) = 25+50x+25x^2+50x^4$, so $5L(1+2x+x^2+2x^4) \neq L(5(1+2x+x^2+2x^4))$.

Students often wonder how to tell at a glance if something is linear or not. In general, if you only end up doing linear combinations with your entries (add, multiply by a constant), then it is probably linear. But if the definition involves multiplication, roots or exponents, then the mapping is probably not linear.

ASSIGNMENT 1n: p.233-4 B1, B2, D3

Solution to Assignment 1n

B1(a) $L(t(a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2)) = L((ta_1+a_2)+(tb_1+b_2)x+(tc_1+c_2)x^2) = \begin{bmatrix} ta_1+a_2 \\ tb_1+b_2 \\ tc_1+c_2 \end{bmatrix} = t\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = tL(a_1+b_1x+c_1x^2) + L(a_2+b_2x+c_2x^2)$

B1(b) $L(t(a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2)) = L((ta_1+a_2)+(tb_1+b_2)x+(tc_1+c_2)x^2) = (tb_1+b_2)+2(tc_1+c_2) = tb_1+2tc_1+b_2+2c_2 = t(b_1+2c_1)+(b_2+2c_2) = tL(a_1+b_1x+c_1x^2)+L(a_2b_2x+c_2x^2)$

B1(c) $L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & tb_1+b_2 \\ tc_1+c_2 & td_1+d_2 \end{bmatrix}\right) = 0 = t(0)+0 = tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

B1(d) $L\left(t\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & 0 \\ 0 & tb_1+b_2 \end{bmatrix}\right) = (ta_1+a_2)+(ta_1+a_2+tb_1+b_2)x+(tb_1+b_2)x^2 = (ta_1+(ta_1+tb_1)x+tb_1x^2)+(a_2+(a_2+b_2)x+b_2x^2) = t(a_1+(a_1+b_1)x+b_1x^2)+(a_2+(a_2+b_2)x+b_2x^2) = tL\left(\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}\right)$

B1(e) $L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & tb_1+b_2 \\ tc_1+c_2 & td_1+d_2 \end{bmatrix}\right) = \begin{bmatrix} (ta_1+a_2)-(tb_1+b_2) & (tb_1+b_2)-(tc_1+c_2) \\ (tc_1+c_2)-(td_1+d_2) & (td_1+d_2)-(ta_1+a_2) \end{bmatrix} = t\begin{bmatrix} a_1-b_1 & b_1-c_1 \\ c_1-d_1 & d_1-a_1 \end{bmatrix} + \begin{bmatrix} a_2-b_2 & b_2-c_2 \\ c_2-d_2 & d_2-a_2 \end{bmatrix} = tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

B1(f) $L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & tb_1+b_2 \\ tc_1+c_2 & td_1+d_2 \end{bmatrix}\right) = (ta_1+a_2+tb_1+b_2+tc_1+c_2+td_1+d_2)x^2 = ta_1x^2+a_2x^2+tb_1x^2+b_2x^2+tc_1x^2+c_2x^2+td_1x^2+d_2x^2 = t(a_1+b_1+c_1+d_1)x^2+(a_2+b_2+c_2+d_2)x^2 = tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

B1(g) Using properties of the transpose (Theorem 3.1.2), we have that $T(tA+B) = (tA+B)^T = (tA)^T + B^T = tA^T + B^T = tT(A)+T(B)$.

B2(a) L is not linear. Consider, for example, that $L(1) = L(1+0x+0x^2+0x^3) = \sqrt{1^2+0^2+0^2+0^2} = 1$ and $L(x) = L(0+x+0x^2+0x^3) = \sqrt{0^2+1^2+0^2+0^2} = 1$, so $L(1)+L(x) = 2$, but $L(1+x) = L(1+x+0x^2+0x^3) = \sqrt{1^2+1^2+0^2+0^2} = \sqrt{2}$, so $L(1)+L(x) \neq L(1+x)$.

B2(b) M is not linear. Consider, for example, that $L(1+2x+3x^2) = 2^2 - 4(1)(3) = -8$ and $L(2+3x+4x^2) = 3^2-4(2)(4) = -23$, so $L(1+2x+3x^2)+$

$L(2+3x+4x^2) = -8-23 = -31$. But $L(1+2x+3x^2+2+3x+4x^2) = L(3+5x+7x^2) = 5^2 - 4(3)(7) = -59$. So $L(1+2x+3x^2) + L(2+3x+4x^2) \neq L(1+2x+3x^2+2+3x+4x^2)$.

B2(c) N is linear. $N\left(t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = N\left(\begin{bmatrix} tx_1+y_1 \\ tx_2+y_2 \\ tx_3+y_3 \end{bmatrix}\right)$ $= \begin{bmatrix} tx_1+y_1-tx_3-y_3 & 0 \\ tx_2+y_2 & tx_2-y_2 \end{bmatrix} = \begin{bmatrix} tx_1-tx_3 & 0 \\ tx_2 & tx_2 \end{bmatrix} + \begin{bmatrix} y_1-y_3 & 0 \\ y_2 & y_2 \end{bmatrix} =$ $t\begin{bmatrix} x_1-x_3 & 0 \\ x_2 & x_2 \end{bmatrix} + \begin{bmatrix} y_1-y_3 & 0 \\ y_2 & y_2 \end{bmatrix} = tN\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + N\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$

B2(d) L is linear. $L(tA+B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}(tA+B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}tA + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}B =$ $t\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}A + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}B = tL(A) + L(B)$

B2(e) T is not linear. Consider, for example, that $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1+2x+6x^2$ and $T\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = 2+6x+24x^2$, so $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = (1+2x+6x^2)+(2+6x+24x^2) = 3+8x+30x^2$. But $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}\right) = 3+15x+105x^2$. And so we see that $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right)$

D3(a) Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Suppose that $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is linearly independent. Consider the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}_{\mathcal{V}}$$

Since $L(\mathbf{0}_{\mathcal{V}}) = \mathbf{0}_{\mathcal{W}}$, we have that $L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k) = \mathbf{0}_{\mathcal{W}}$. But, using the linearity properties of L, this means that

$$t_1L(\mathbf{v}_1) + \cdots + t_kL(\mathbf{v}_k) = \mathbf{0}_{\mathcal{W}}$$

Since the set $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is linearly independent, the only solution to this equation is $t_1 = \cdots = t_k = 0$. This means we have shown that the only solution to the equation $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}_{\mathcal{V}}$ is $t_1 = \cdots = t_k = 0$. Which means that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

D3(b) There are many examples of this. One easy thing to do is to map one of your vectors to the zero vector, since any set containing the zero vector is linearly dependent. For example, let $\mathbb{V} = \mathbb{W} = \mathbb{R}^3$, and let $L : \mathbb{V} \to \mathbb{W}$ be defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$. Then L is linear, since $L\left(t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} tx_1 + y_1 \\ tx_2 + y_2 \\ tx_3 + y_3 \end{bmatrix}\right) = \begin{bmatrix} tx_1 + y_1 \\ tx_2 + y_2 \\ 0 \end{bmatrix} = t\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = tL\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$. We note that the set $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is linearly independent, but the set $\left\{L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ is linearly dependent.