Lecture 1m

The Change of Coordinates Matrix

(pages 221-224)

At the end of the previous lecture, we found the coordinates of $p(x) = 6-2x+2x^2$ with respect to two different bases. It happens that sometimes you will start with the coordinates for a vector with respect to one basis, but want to get the coordinates of the vector with respect to another basis.

Example: Let $\mathcal{B} = \{1+x-x^2, x+x^2, -x+3x^2\}$ and $\mathcal{C} = \{1+x+x^2, 1-x-2x^2, 4x\}$, and let $p(x) \in P_2$ be such that $[p(x)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Find $[p(x)]_{\mathcal{B}}$.

The first step to finding $[p(x)]_{\mathcal{B}}$ is to find what $p(x)$ is. Using the given $\mathcal{C}$-coordinates of $p(x)$, this is a straightforward calculation:

$$\begin{aligned} p(x) &= 3(1+x+x^2) + 2(1-x-2x^2) + (4x) \\ &= 5 + 5x - x^2 \end{aligned}$$

Now we simply need to find the $\mathcal{B}$-coordinates of $5+5x-x^2$. That is, we need to find scalars t_1, t_2, and t_3 such that

$$5+5x-x^2 = t_1(1+x-x^2)+t_2(x+x^2)+t_3(-x+3x^2) = (t_1)+(t_1+t_2-t_3)x+(-t_1+t_2+3t_3)x^2$$

Setting the coefficients equal to each other, we see that we are looking for the solution to the following system:

$$\begin{array}{rrrl} t_1 & & & = 5 \\ t_1 & +t_2 & -t_3 & = 5 \\ -t_1 & +t_2 & +3t_3 & = -1 \end{array}$$

To find the solution, we will row reduce its augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 1 & 1 & -1 & 5 \\ -1 & 1 & 3 & -1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 4 \end{array}\right] R_3 - R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 4 \end{array}\right] (1/4)R_3 \quad \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] R_2 + R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

And so we see that $t_1 = 5$, $t_2 = 1$, and $t_3 = 1$. And this means that $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$.

In the previous example, we could replace the $\mathcal{C}$-coordinates with any vector from $\mathbb{R}^3$, and the exact same steps would lead us to the $\mathcal{B}$-coordinates. But instead of doing these steps over and over, we can actually pack all of this information into a single matrix that we multiply our coordinate vector by. In order to find this matrix, we first need to note the following fact.

Theorem 4.4.1: Let $\mathcal{B}$ be a basis for a finite dimensional vector space $\mathbb{V}$. Then, for any $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $t \in \mathbb{R}$, we have

$$[t\mathbf{x} + \mathbf{y}]_{\mathcal{B}} = t[x]_{\mathcal{B}} + [y]_{\mathcal{B}}$$

Proof of Theorem 4.4.1: Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, let $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, and let $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Then we have $\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$ and $\mathbf{y} = y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n$, so we get

$$\begin{aligned} t\mathbf{x} + \mathbf{y} &= t(x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n) + (y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n) \\ &= (tx_1\mathbf{v}_1 + \cdots + tx_n\mathbf{v}_n) + (y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n) \\ &= (tx_1 + y_1)\mathbf{v}_1 + \cdots + (tx_n + y_n)\mathbf{v}_n \end{aligned}$$

This means that the $\mathcal{B}$-coordinates for $t\mathbf{x} + \mathbf{y}$ are $\begin{bmatrix} tx_1 + y_1 \\ \vdots \\ tx_n + y_n \end{bmatrix}$. Which means that

$$
\begin{aligned}
[t\mathbf{x}+\mathbf{y}]_{\mathcal{B}} &= \begin{bmatrix} tx_1+y_1 \\ \vdots \\ tx_n+y_n \end{bmatrix} \\
&= \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
&= t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
&= t[x]_{\mathcal{B}} + [y]_{\mathcal{B}}
\end{aligned}
$$

So how does this Theorem help us? Well, suppose we have two bases $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for a vector space $\mathbb{V}$, and let $\mathbf{x} \in \mathbb{V}$ be such that $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. To find $[\mathbf{x}]_{\mathcal{B}}$ we use Theorem 4.4.1 to note the following:

$$
\begin{aligned}
[\mathbf{x}]_{\mathcal{B}} &= [x_1\mathbf{w}_1 + \cdots + x_n\mathbf{w}_n]_{\mathcal{B}} \\
&= x_1[\mathbf{w}_1]_{\mathcal{B}} + \cdots + x_n[\mathbf{w}_n]_{\mathcal{B}} \\
&= \begin{bmatrix} [\mathbf{w}_1]_{\mathcal{B}} & \cdots & [\mathbf{w}_n]_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} [\mathbf{w}_1]_{\mathcal{B}} & \cdots & [\mathbf{w}_n]_{\mathcal{B}} \end{bmatrix} [\mathbf{x}]_{\mathcal{C}}
\end{aligned}
$$

This means that to find the $\mathcal{B}$-coordinates for $\mathbf{x}$, we can multiply the $\mathcal{C}$-coordinates by a matrix whose columns are the $\mathcal{B}$-coordinates of the vectors in $\mathcal{C}$. This leads us to the following definition.

Definition: Let $\mathcal{B}$ and $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ both be bases for a vector space $\mathbb{V}$. The matrix $P = \begin{bmatrix} [\mathbf{w}_1]_{\mathcal{B}} & \cdots & [\mathbf{w}_n]_{\mathcal{B}} \end{bmatrix}$ is called the **change of coordinates matrix** from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates, and satisfies

$$[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}}$$

Example: Let's continue our example from before, and find the change of coordinates matrix from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates. To do this, we need to find $[1+x+x^2]_{\mathcal{B}}$, $[1-x-2x^2]_{\mathcal{B}}$, and$[4x]_{\mathcal{B}}$. That is, we need to find scalars a_1, a_2, and a_3 such that

$$1+x+x^2 = a_1(1+x-x^2)+a_2(x+x^2)+a_3(-x+3x^2) = (a_1)+(a_1+a_2-a_3)x+(-a_1+a_2+3a_3)x^2$$

which is equivalent to the system

$$\begin{array}{rrrl} a_1 & & & = 1 \\ a_1 & +a_2 & -a_3 & = 1 \\ -a_1 & +a_2 & +3a_3 & = 1 \end{array}$$

Before we find the solution to this system, let's go ahead and set up the systems for our other two basis vectors. For the second $\mathcal{C}$ polynomial, we need to find scalars b_1, b_2, and b_3 such that

$$1-x-2x^2 = b_1(1+x-x^2)+b_2(x+x^2)+b_3(-x+3x^2) = (b_1)+(b_1+b_2-b_3)x+(-b_1+b_2+3b_3)x^2$$

which is equivalent to the system

$$\begin{array}{rrrl} b_1 & & & = 1 \\ b_1 & +b_2 & -b_3 & = -1 \\ -b_1 & +b_2 & +3b_3 & = -2 \end{array}$$

For the third polynomial in $\mathcal{C}$, we need to find scalars c_1, c_2, and c_3 such that

$$4x = c_1(1+x-x^2)+c_2(x+x^2)+c_3(-x+3x^2) = (c_1)+(c_1+c_2-c_3)x+(-c_1+c_2+3c_3)x^2$$

which is equivalent to the system

$$\begin{array}{rrrl} c_1 & & & = 0 \\ c_1 & +c_2 & -c_3 & = 4 \\ -c_1 & +c_2 & +3c_3 & = 0 \end{array}$$

Now, all three of these systems have the same coefficient matrix, so we can solve them simultaneously by row reducing the following triply augmented matrix:

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & -1 & 4 \\ -1 & 1 & 3 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + R_1 \end{array} \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 4 \\ 0 & 1 & 3 & 2 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 4 \\ 0 & 0 & 4 & 2 & 1 & -4 \end{array}\right] \begin{array}{l} \\ \\ (1/4)R_3 \end{array} \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 1/2 & 1/4 & -1 \end{array}\right] \begin{array}{l} \\ R_2 + R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & -7/4 & 3 \\ 0 & 0 & 1 & 1/2 & 1/4 & -1 \end{array}\right]$$

Reading off the first augmented column, we see that $a_1 = 1$, $a_2 = 1/2$, and $a_3 = 1/2$, so $[1+x+x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix}$. Reading off the second augmented column, we see that $b_1 = 1$, $b_2 = -7/4$, and $b_3 = 1/4$, so $[1 - x - 2x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -7/4 \\ 1/4 \end{bmatrix}$. And reading off the third augmented column, we see that $c_1 = 0$, $c_2 = 3$, and $c_3 = -1$, so $[4x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$. And this means that our change of coordinates matrix P is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1/2 & -7/4 & 3 \\ 1/2 & 1/4 & -1 \end{bmatrix}$$

Notice that P is the same as the right side of our RREF matrix above. Note also that $P\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1/2 & -7/4 & 3 \\ 1/2 & 1/4 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+2+0 \\ 3/2 - 7/2 + 3 \\ 3/2 + 1/2 - 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$, which is the same result we got in the original example.

Theorem 4.4.2: Let $\mathcal{B}$ and $\mathcal{C}$ both be bases for a finite-dimensional vector space $\mathbb{V}$. Let P be the change of coordinates matrix from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates. Then P is invertible and P^{-1} is the change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

Proof of Theorem 4.4.2: To see that P^{-1} is the change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates, note that

$$P^{-1}[\mathbf{x}]_{\mathcal{B}} = P^{-1}(P[\mathbf{x}]_{\mathcal{C}}) = (P^{-1}P)[\mathbf{x}]_{\mathcal{C}} = I[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{C}}$$

Example: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be the standard basis for $M(2,2)$, and let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} \right\}$. Find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates, and find the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates.

The change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates is

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}_{\mathcal{S}} \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}_{\mathcal{S}} \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix}_{\mathcal{S}} \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix}_{\mathcal{S}} \right]$$

But we can find these coordinates without any calculations:

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} = -1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} = -1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 3 \\ 2 \\ 8 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 7 \end{bmatrix}$$

and so we see that

$$Q = \begin{bmatrix} 1 & -1 & 3 & -1 \\ 2 & 0 & 2 & 4 \\ 3 & -1 & 8 & 1 \\ 1 & 2 & -3 & 7 \end{bmatrix}$$

To find the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates, we use Theorem 4.4.2, which tells us that $P = Q^{-1}$, and then we use the matrix inverse algorithm to find Q^{-1}:

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & -1 & 8 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -3 & 7 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array}$$
$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -4 & 6 & -2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 4 & -3 & 0 & 1 & 0 \\ 0 & 3 & -6 & 8 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (1/2)R_2 \\ \\ \\ \end{array}$$
$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & -1 & 1/2 & 0 & 0 \\ 0 & 2 & -1 & 4 & -3 & 0 & 1 & 0 \\ 0 & 3 & -6 & 8 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & 3 & -2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -3/2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ (-1)R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & 3 & -2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 3/2 & 0 & -1 \end{array}\right] \begin{array}{l} R_1 + R_4 \\ R_2 - 3R_4 \\ R_3 + 2R_4 \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & -1 & 3/2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 5 & -4 & 0 & 3 \\ 0 & 0 & 3 & 0 & -5 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3/2 & 0 & -1 \end{array}\right] \begin{array}{l} \\ \\ (1/3)R_3 \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & -1 & 3/2 & 0 & -1 \\ 0 & 1 & -2 & 0 & 5 & -4 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5/3 & 2/3 & 1/3 & -2/3 \\ 0 & 0 & 0 & 1 & -2 & 3/2 & 0 & -1 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_2 + 2R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 4 & -1/2 & -1 & 1 \\ 0 & 1 & 0 & 0 & 5/3 & -8/3 & 2/3 & 5/3 \\ 0 & 0 & 1 & 0 & -5/3 & 2/3 & 1/3 & -2/3 \\ 0 & 0 & 0 & 1 & -2 & 3/2 & 0 & -1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 17/3 & -19/6 & -1/3 & 8/3 \\ 0 & 1 & 0 & 0 & 5/3 & -8/3 & 2/3 & 5/3 \\ 0 & 0 & 1 & 0 & -5/3 & 2/3 & 1/3 & -2/3 \\ 0 & 0 & 0 & 1 & -2 & 3/2 & 0 & -1 \end{array}\right]$$

And so we see that $P = Q^{-1} = \begin{bmatrix} 17/3 & -19/6 & -1/3 & 8/3 \\ 5/3 & -8/3 & 2/3 & 5/3 \\ -5/3 & 2/3 & 1/3 & -2/3 \\ -2 & 3/2 & 0 & -1 \end{bmatrix}$