Lecture 1l

Coordinates

(pages 218-221)

Once we have a basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for a vector space $\mathbb{V}$, the Unique Representation Theorem tells us that for any given vector $\mathbf{x} \in \mathbb{V}$, there is a unique way to write $\mathbf{x}$ as a linear combination of the vectors in $\mathcal{B}$. That is, there is a unique collection of scalars $x_1, \dots, x_n \in \mathbb{R}$ such that

$$\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$$

This leads to a situation similar to the one that lead to the creation of matrices. That is, if we take the basis $\mathcal{B}$ as a given, do we really need to write the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ every time? The real information is contained in the list of scalars, so let's just write those. Since we are only looking at a single list of numbers, this is best represented as a vector from $\mathbb{R}^n$.

Definition: Suppose that $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for the vector space $\mathbb{V}$. If $\mathbf{x} \in \mathbb{V}$ with $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$, then the **coordinate vector** of $\mathbf{x}$ with respect to the basis $\mathcal{B}$ is

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We also refer to $[\mathbf{x}]_{\mathcal{B}}$ as "the coordinates of $\mathbf{x}$ with respect to $\mathcal{B}$", or "the $\mathcal{B}$-coordinates of $\mathbf{x}$."

The easiest examples of coordinates are those we get using the standard bases of our common vector spaces. Easiest of all is when we use the standard basis for $\mathbb{R}^n$ to find the coordinates of a vector in $\mathbb{R}^n$, since they are the same.

Example: Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis for $\mathbb{R}^3$. If $\mathbf{x} = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$, then we have that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$$

since $\begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. In general, if $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then we have that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

since $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Things get at least a little more interesting when we look at matrix spaces or polynomial spaces.

Example: Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be the standard basis for $M(2,2)$. If $\mathbf{x} = \begin{bmatrix} 3 & -9 \\ -8 & 2 \end{bmatrix}$, then

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -9 \\ -8 \\ 2 \end{bmatrix}$$

since

$$\begin{bmatrix} 3 & -9 \\ -8 & 2 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 9\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 8\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Let $\mathcal{B} = \{1, x, x^2, x^3, x^4, x^5\}$ be the standard basis for P_5. If $\mathbf{x} = 5 - 3x + 7x^2 + x^4 - 8x^5$, then

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -3 \\ 7 \\ 0 \\ 1 \\ -8 \end{bmatrix}$$

since $5 - 3x + 7x^2 + x^4 - 8x^5 = 5(1) - 3(x) + 7(x^2) + 0(x^3) + 1(x^4) - 8(x^5)$.

Of course, things get really interesting when we use something that is not a standard basis.

Example: In Lecture 1h, we showed that

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} \right\}$$

is a basis for $M(2,2)$. If we note that

$$2\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + 5\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -10 & 4 \\ -14 & 25 \end{bmatrix}$$

then we see that

$$\begin{bmatrix} -10 & 4 \\ -14 & 25 \end{bmatrix}_{\mathcal{A}} = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

But usually we start with a vector, and want to find it's coordinates with respect to our basis. To that end, let's try to find the $\mathcal{A}$ coordinates for $\begin{bmatrix} -7 & 18 \\ 2 & 38 \end{bmatrix}$. That is, we want to find scalars x_1, x_2, x_3, and x_4 such that

$$x_1\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + x_2\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} + x_3\begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix} + x_4\begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 18 \\ 2 & 38 \end{bmatrix}$$

Performing the calculation on the left, this becomes

$$\begin{bmatrix} x_1 - x_2 + 3x_3 - x_4 & 2x_1 + 2x_3 + 4x_4 \\ 3x_1 - x_2 + 8x_3 + x_4 & x_1 + 2x_2 - 3x_3 + 7x_4 \end{bmatrix} = \begin{bmatrix} -7 & 18 \\ 2 & 38 \end{bmatrix}$$

Setting the entries equal, we get the following system of linear equations:

$$\begin{array}{rrrrl} x_1 & -x_2 & +3x_3 & -x_4 & = -7 \\ 2x_1 & & +2x_3 & +4x_4 & = 18 \\ 3x_1 & -x_2 & +8x_3 & +x_4 & = 2 \\ x_1 & +2x_2 & -3x_3 & +7x_4 & = 38 \end{array}$$

To solve this system, we will row reduce its augmented matrix as follows:

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & -7 \\ 2 & 0 & 2 & 4 & 18 \\ 3 & -1 & 8 & 1 & 2 \\ 1 & 2 & -3 & 7 & 38 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & -7 \\ 0 & 2 & -4 & 6 & 32 \\ 0 & 2 & -1 & 4 & 23 \\ 0 & 3 & -6 & 8 & 45 \end{array}\right] \begin{array}{l} \\ (1/2)R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & -7 \\ 0 & 1 & -2 & 3 & 16 \\ 0 & 2 & -1 & 4 & 23 \\ 0 & 3 & -6 & 8 & 45 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & -7 \\ 0 & 1 & -2 & 3 & 16 \\ 0 & 0 & 3 & -2 & -9 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right] \begin{array}{l} \\ \\ \\ (-1)R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & -1 & -7 \\ 0 & 1 & -2 & 3 & 16 \\ 0 & 0 & 3 & -2 & -9 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 + R_4 \\ R_2 - 3R_4 \\ R_3 + 2R_4 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 7 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \\ \\ (1/3)R_3 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_1 + 2R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

And so, from our RREF matrix, we see that the solution is $x_1 = 4$, $x_2 = 5$, $x_3 = -1$, $x_4 = 3$. This means that

$$4\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + 5\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix} + 3\begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 18 \\ 2 & 38 \end{bmatrix}$$

and so we have

$$\begin{bmatrix} -7 & 18 \\ 2 & 38 \end{bmatrix}_{\mathcal{A}} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 3 \end{bmatrix}$$

One last thing to note in regards to coordinates is that the order of the basis vectors matters. That is to say, the set $\{1, x, x^2\}$ is not the same basis for P_2 as the set $\{x, x^2, 1\}$, simply because the order we wrote the basis vectors in changed. And this change results in different coordinates for our vectors.

Example: Let $\mathcal{E} = \{1, x, x^2\}$ be the standard basis for P_2, let $\mathcal{B} = \{x, x^2, 1\}$ be another basis for P_2, and let $p(x) = 6 - 2x + 2x^2$. Then we have that $[p(x)]_{\mathcal{E}} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$, since $p(x) = 6(1) - 2(x) + 2(x^2)$, while $[p(x)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$, since $p(x) = -2(x) + 2(x^2) + 6(1)$.