Solution to Practice 1k

B7(a) We know that a plane in $\mathbb{R}^3$ has dimension 2. So, we just need to find 2 linearly independent vectors in the plane. By inspection, we see that $(3)+3(-1)+4(0)=0$, so $\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ lies in the plane. We also see that $(4)+3(0)+4(-1)=0$, so $\vec{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$ also lies in the plane. And since $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$, we know that the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. So a basis for the plane $x_1+3x_2+4x_3=0$ is the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \right\}$.

B7(b) To extend our basis from part (a) to a basis for all of $\mathbb{R}^3$, we need to find a third vector that is not in the span of our basis for the plane. But the span of the basis for the plane is exactly the plane, so we simply need to find a vector that is not in the plane. By inspection, we note that $(1)+3(0)+4(0)\neq 0$, so the vector $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the plane. And thus, the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

B8(a) We know that a hyperplane in $\mathbb{R}^4$ has dimension 3. So, we need to find 3 linearly independent vectors in the hyperplane. By inspections, we see that $(1)+(-1)+2(0)+(0)=0$, $(1)+(0)+2(0)+(-1)=0$, and $(1)+(1)+2(-1)+(0)=0$, so $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$ are three vectors in the hyperplane. To see if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, we need to see if there are any non-trivial solutions to the equation

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

This is equivalent to the following homogeneous system of equations:

$$\begin{array}{rrrl} t_1 & +t_2 & +t_3 & = 0 \\ -t_1 & & t_3 & = 0 \\ & & -2t_3 & = 0 \\ & -t_2 & & = 0 \end{array}$$

The fourth equation tells us that $t_2 = 0$, and the third equation tells us that $t_3 = 0$. Plugging these values into the first equation, we get $t_1 + 0 + 0 = 0$, which means that $t_1 = 0$. As such, the only solution to this system is $t_1 = t_2 = t_3 = 0$, which means that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, and thus a basis for the hyperplane.

B8(b) To extend our basis from part (a) to a basis for all of $\mathbb{R}^4$, we need to find a fourth vector that is not in the span of our basis for the hyperplane. But the span of the basis for the hyperplane is exactly the hyperplane, so we simply need to find a vector that is not in the hyperplane. By inspection, we note that $(1) + (1) + 2(1) + (1) \neq 0$, so the vector $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the hyperplane. And thus, the set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

D3(a): Let $\mathbb{V}$ be a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2 + t\mathbf{v}_1\}$ for some $t \in \mathbb{R}$. We want to show that $\mathcal{B}$ is also a basis for $\mathbb{V}$. The first thing we want to notice is that $\mathbf{v}_1 \neq \mathbf{v}_2 + t\mathbf{v}_1$. To see this, assume by way of contradiction that $\mathbf{v}_1 = \mathbf{v}_2 + t\mathbf{v}_1$. Then $\mathbf{v}_1 - \mathbf{v}_1 = \mathbf{v}_2 + t\mathbf{v}_1 - \mathbf{v}_1$, so $\mathbf{0} = \mathbf{v}_2 + (t-1)\mathbf{v}_1$. But this means that there is a solution other than $t_1 = t_2 = 0$ to the equation $\mathbf{0} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$, which contradicts the fact that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{V}$, and thus linearly independent.

So, we know that $\mathcal{B}$ has two vectors. By the two-out-of-three rule, to prove that $\mathcal{B}$ is a basis for $\mathbb{V}$, it will suffice to show that $\mathcal{B}$ is a spanning set for $\mathbb{V}$. To that end, let $\mathbf{x} \in \mathbb{V}$. Then, since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{V}$ (and thus a spanning set for $\mathbb{V}$), we know that there are scalars s_1 and s_2 such that $\mathbf{x} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$. But this means that

$$\begin{aligned} \mathbf{x} &= s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \\ &= s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \mathbf{0} \\ &= s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + s_2t\mathbf{v}_1 - s_2t\mathbf{v}_1 \\ &= (s_1\mathbf{v}_1 - s_2t\mathbf{v}_1) + (s_2\mathbf{v}_2 + s_2t\mathbf{v}_1) \\ &= (s_1 - s_2t)\mathbf{v}_1 + s_2(\mathbf{v}_2 + t\mathbf{v}_1) \end{aligned}$$

So, if we let $t_1 = s_1 - s_2 t$ and $t_2 = s_2$, then we have found scalars $t_1, t_2 \in \mathbb{R}$ such that $\mathbf{x} = t_1\mathbf{v}_1 + t_2(\mathbf{v}_2 + t\mathbf{v}_1)$, which means that $\mathbf{x} \in \text{Span } \mathcal{B}$. Since every $\mathbf{x} \in \mathbb{V}$ is in Span $\mathcal{B}$, we know that $\mathcal{B}$ is a spanning set for $\mathbb{V}$. Which shows that $\mathcal{B}$ is a basis for $\mathbb{V}$.