Solution to Practice 1i

B4(a): Since we already know that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$, we want to check if $\mathcal{B}$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 - t_3 \\ 2t_1 + 3t_2 \\ t_1 - 2t_2 + 7t_3 \end{bmatrix}$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrrl} t_1 & +2t_2 & -t_3 & = 0 \\ 2t_1 & +3t_2 & & = 0 \\ t_1 & -2t_2 & +7t_3 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & -2 & 7 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & -4 & 8 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 4R_2 \end{array} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 2. Since this is less than the number of variables, we know that our equation has an infinite number of solutions, which means that $\mathcal{B}$ is linearly dependent. This means that $\mathcal{B}$ is not a basis for Span $\mathcal{B}$. So we need to remove at least one dependent member of $\mathcal{B}$. To find such a dependent member, we will find the general solution to our equation. And the first step in that process will be to complete the row reduction of our coefficient matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ -R_2 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

From our RREF matrix, we see that our system is equivalent to the system

$$\begin{array}{rrrl} t_1 & & +3t_3 & = 0 \\ & t_2 & -2t_3 & = 0 \end{array}$$

Replacing the variable t_3 with the parameter t, we see that the general solution to our system is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} -3t \\ 2t \\ t \end{bmatrix}$$

Plugging in $t = -1$, we get the solution $t_1 = 3$, $t_2 = -2$, and $t_3 = -1$, which translates into the following equation:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

Since we can write $\begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix}$ as a linear combination of the other elements of $\mathcal{B}$, we know that $\mathcal{B}_1 = \mathcal{B} \setminus \left\{ \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \right\}$ is also a spanning set for $\mathcal{B}$. So now we want to see if $\mathcal{B}_1$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 \\ 2t_1 + 3t_2 \\ t_1 - 2t_2 \end{bmatrix}$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrl} t_1 & +2t_2 & = 0 \\ 2t_1 & +3t_2 & = 0 \\ t_1 & -2t_2 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -4 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 4R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 2. Since this is the same as the number of variables, we know that our equation has a unique solution, which means that $\mathcal{B}_1$ is linearly independent. And since $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \right\}$ is a linearly independent spanning set for Span $\mathcal{B}$, we see that $\mathcal{B}_1$ is a basis for Span $\mathcal{B}$.

B4(b) Since we already know that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$, we want to check if $\mathcal{B}$ is linearly independent. To that end, we will look for solutions to the equation

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + t_5 \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}
$$

$$
= \begin{bmatrix} t_1 + t_2 + 2t_3 + 3t_5 \\ t_1 - t_2 + 2t_4 - 3t_5 \\ -t_1 + 2t_2 + t_3 - 3t_4 - 2t_5 \end{bmatrix}
$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$
\begin{array}{rrrrrl}
t_1 & +t_2 & +2t_3 & & +3t_5 & = 0 \\
t_1 & -t_2 & & +2t_4 & -3t_5 & = 0 \\
-t_1 & +2t_2 & +t_3 & -3t_4 & -2t_5 & = 0
\end{array}
$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$
\begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 1 & -1 & 0 & 2 & -3 \\ -1 & 2 & 1 & -3 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & -2 & -2 & 2 & -6 \\ 0 & 3 & 3 & -3 & 1 \end{bmatrix} \begin{matrix} \\ (-1/2)R_2 \\ \\ \end{matrix}
$$

$$
\sim \begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 3 & 3 & -3 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}
$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 2. Since this is less than the number of variables, we know that our equation has an infinite number of solutions, which means that $\mathcal{B}$ is linearly dependent. This means that $\mathcal{B}$ is not a basis for Span $\mathcal{B}$. So we need to remove at least one dependent member of $\mathcal{B}$. To find such a dependent member, we will find the general solution to our equation. And the first step in that process will be to complete the row reduction of our coefficient matrix:

$$
\begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \\ (1/2)R_3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 3R_3 \\ R_2 - 3R_3 \\ \\ \end{matrix}
$$

$$
\sim \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

From our RREF matrix, we see that our system is equivalent to the system

$$
\begin{array}{rl}
t_1 + t_3 + t_4 & = 0 \\
t_2 + t_3 - t_4 & = 0 \\
t_5 & = 0
\end{array}
$$

Replacing the variable t_3 with the parameter s and the variable t_4 with the parameter t, we see that the general solution to our equation is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ -s+t \\ s \\ t \\ 0 \end{bmatrix}$$

Setting $s=-1$ and $t=0$, we see that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$, which means that $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. And this means that $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is a dependent member of $\mathcal{B}$.

But notice also that setting $t=-1$ and $s=0$ gives us $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$
which means that $\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. And this means that $\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ is also a dependent member of $\mathcal{B}$.

Which one should we remove? Well, both actually. The easiest way to see this is to realize that we can first remove either one. Let's go ahead and remove $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, leaving us with $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix} \right\}$. By Theorem A we know that $\mathcal{B}_1$ is a spanning set for Span $\mathcal{B}$. But since we already know that $\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, we know that $\mathcal{B}_1$ is not linearly independent. So let's now remove the dependent member $\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$, and consider the set $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix} \right\}$. By Theorem A, we know that $\mathcal{B}_2$ is a spanning set for Span $\mathcal{B}$. Now let's check to see if $\mathcal{B}_2$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 + 3t_5 \\ t_1 - t_2 - 3t_5 \\ -t_1 + 2t_2 - 2t_5 \end{bmatrix}$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrrl} t_1 & +t_2 & +3t_5 & = 0 \\ t_1 & -t_2 & -3t_5 & = 0 \\ -t_1 & +2t_2 & -2t_5 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -6 \\ 0 & 3 & 1 \end{bmatrix} (-1/2)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} R_3 - 3R_2 \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 3. Since this equals the number of variables, we know that our equation has a unique solution, which means that $\mathcal{B}_2$ is linearly independent. This means that $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix} \right\}$ is a basis for Span $\mathcal{B}$.

B5(a) Since we already know that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$, we want to check if $\mathcal{B}$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= t_1 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} + t_3 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + t_4 \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} t_1 - 2t_2 + t_3 + 3t_4 & t_1 - 2t_2 - 1t_3 - 1t_4 \\ 2t_1 - 4t_2 + t_3 + 5t_4 & t_1 - 2t_2 + 2t_4 \end{bmatrix} \end{aligned}$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrrrl} t_1 & -2t_2 & +t_3 & +3t_4 & = 0 \\ t_1 & -2t_2 & -t_3 & +t_4 & = 0 \\ 2t_1 & -4t_2 & +t_3 & +5t_4 & = 0 \\ t_1 & -2t_2 & & +2t_4 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ 2 & -4 & 1 & 5 \\ 1 & -2 & 0 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - (1/2)R_2 \\ R_4 - (1/2)R_2 \end{matrix} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 2. Since this is less than the number of variables, we know that our equation has an infinite number of solutions, which means that $\mathcal{B}$ is linearly dependent. This means that $\mathcal{B}$ is not a basis for Span $\mathcal{B}$. So we need to remove at least one dependent member of $\mathcal{B}$. To find such a dependent member, we will find the general solution to our equation. And the first step in that process will be to complete the row reduction of our coefficient matrix:

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ (-1/2)R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From our RREF matrix, we see that our system is equivalent to the system

$$\begin{aligned} t_1 - 2t_2 + 2t_4 &= 0 \\ t_3 + t_4 &= 0 \end{aligned}$$

Replacing the variable t_2 with the parameter s and the variable t_4 with the parameter t, we see that the general solution to our equation is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 2s - 2t \\ s \\ -t \\ t \end{bmatrix}$$

Setting $s = -1$ and $t = 0$, we see that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} + 0\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, which means that $\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. And this means that $\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$ is a dependent member of $\mathcal{B}$.

But notice also that setting $t = -1$ and $s = 0$ gives us $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + 0\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, which means that $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

Which one should we remove? Well, both actually. The easiest way to see this is to realize that we can first remove either one. Let's go ahead and remove $\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$, leaving us with $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \right\}$. By Theorem A we know that $\mathcal{B}_1$ is a spanning set for Span $\mathcal{B}$. But since we already know that $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, we know that $\mathcal{B}_1$ is not linearly independent. So let's now remove the dependent member $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, and consider the set $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right\}$. By Theorem A, we know that $\mathcal{B}_2$ is a spanning set for Span $\mathcal{B}$. Now let's check to see if $\mathcal{B}_2$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 & t_1 - t_2 \\ 2t_1 + t_2 & t_1 \end{bmatrix}$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrl} t_1 & +t_2 & = 0 \\ t_1 & -t_2 & = 0 \\ 2t_1 & +t_2 & = 0 \\ t_1 & & = 0 \end{array}$$

The last equation tells us the $t_1 = 0$. Plugging this into the first equation, we see that $0 + t_2 = 0$, so we also have $t_2 = 0$. And since $t_1 = t_2 = 0$ is the only solution to the equation, we see that $\mathcal{B}_2$ is linearly independent. And this means that $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for Span $\mathcal{B}$.

B5(b) Since we already know that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$, we want to check if $\mathcal{B}$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= t_1 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} + t_4 \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + t_5 \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} t_1 + t_2 + 3t_3 + t_5 & 2t_1 + 2t_2 + 6t_3 + t_4 + 3t_5 \\ -t_1 + 2t_2 + 2t_5 & t_1 + t_2 + 3t_3 - 2t_4 - t_5 \end{bmatrix} \end{aligned}$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrrrrl} t_1 & +t_2 & +3t_3 & & +t_5 & =0 \\ 2t_1 & +2t_2 & +6t_3 & +t_4 & +3t_5 & =0 \\ -t_1 & +2t_2 & & & +2t_5 & =0 \\ t_1 & +t_2 & +3t_3 & -2t_4 & -t_5 & =0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\left[\begin{array}{rrrrr} 1 & 1 & 3 & 0 & 1 \\ 2 & 2 & 6 & 1 & 3 \\ -1 & 2 & 0 & 0 & 2 \\ 1 & 1 & 3 & -2 & -1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{rrrrr} 1 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 3 & 3 & 0 & 3 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] R_1 \updownarrow R_2$$

$$\sim \left[\begin{array}{rrrrr} 1 & 1 & 3 & 0 & 1 \\ 0 & 3 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} \\ (1/3)R_2 \\ \\ R_4 + 2R_3 \end{array} \sim \left[\begin{array}{rrrrr} 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 3. Since this is less than the number of variables, we know that our equation has an infinite number of solutions, which means that $\mathcal{B}$ is linearly dependent. This means that $\mathcal{B}$ is not a basis for Span $\mathcal{B}$. So we need to remove at least one dependent member of $\mathcal{B}$. To find such a dependent member, we will find the general solution to our equation. And the first step in that process will be to complete the row reduction of our coefficient matrix:

$$\left[\begin{array}{rrrrr} 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{rrrrr} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From our RREF matrix, we see that our system is equivalent to the system

$$\begin{array}{rl} t_1 + 2t_3 & =0 \\ t_2 + t_3 + t_5 & =0 \\ t_4 + t_5 & =0 \end{array}$$

Replacing the variable t_3 with the parameter s and the variable t_5 with the parameter t, we see that the general solution to our equation is

$$\left[\begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array}\right] = \left[\begin{array}{r} -2s \\ -s-t \\ s \\ -t \\ t \end{array}\right]$$

Setting $s = -1$ and $t = 0$, we see that $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] = 2\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right] + \left[\begin{array}{rr} 1 & 2 \\ 2 & 1 \end{array}\right] -$

$\begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + 0 \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, which means that $\begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. And this means that $\begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}$ is a dependent member of $\mathcal{B}$.

But notice also that setting $t = -1$ and $s = 0$ gives us $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + 0 \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, which means that $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$. And this means that $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ is a dependent member of $\mathcal{B}$.

Which one should we remove? Well, both actually. The easiest way to see this is to realize that we can first remove either one. Let's go ahead and remove $\begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}$, leaving us with $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \right\}$. By Theorem A we know that $\mathcal{B}_1$ is a spanning set for Span $\mathcal{B}$. But since we already know that $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$, we know that $\mathcal{B}_1$ is not linearly independent. So let's now remove the dependent member $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, and consider the set $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$. By Theorem A, we know that $\mathcal{B}_2$ is a spanning set for Span $\mathcal{B}$. Now let's check to see if $\mathcal{B}_2$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 & 2t_1 + 2t_2 + t_3 \\ -t_1 + 2t_2 & t_1 + t_2 - 2t_3 \end{bmatrix}$$

Setting the components equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrrl} t_1 & +t_2 & & = 0 \\ 2t_1 & +2t_2 & +t_3 & = 0 \\ -t_1 & +2t_2 & & = 0 \\ t_1 & +t_2 & -2t_3 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad R_1 \updownarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ (1/3)R_2 \\ \\ R_4 + 2R_3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 3. Since this equals the number of variables, we know that our equation has a unique solution, which means that $\mathcal{B}_2$ is linearly independent. This means that $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$ is a basis for Span $\mathcal{B}$.

B6(a) Since we already know that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$, we want to check if $\mathcal{B}$ is linearly independent. To that end, we will look for solutions to the equation

$$0+0x+0x^2 = t_1(1+x+x^2)+t_2(x+x^2+x^3)+t_3(1-x^3) = (t_1+t_3)+(t_1+t_2)x+(t_1+t_2)x^2+(t_2-t_3)x^3$$

Setting the coefficients equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{aligned} t_1 \quad\quad\; + t_3 &= 0 \\ t_1 + t_2 \quad\quad\; &= 0 \\ t_1 + t_2 \quad\quad\; &= 0 \\ t_2 - t_3 &= 0 \end{aligned}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced row echelon form, and from it we see that the rank of the coefficient matrix is 2. Since this is less than the number of variables, we know that our equation has an infinite number of solutions, which means that $\mathcal{B}$ is linearly dependent. This means that $\mathcal{B}$ is not a basis for Span $\mathcal{B}$. So we need to remove at least one dependent member of $\mathcal{B}$. To find such a dependent member, we will find the general solution to our equation. From our RREF matrix, we see that our system is equivalent to the system

$$\begin{aligned} t_1 + t_3 &= 0 \\ t_2 - t_3 &= 0 \end{aligned}$$

Replacing the variable t_3 with the parameter t, we see that the general solution to our equation is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Setting $t = -1$ gives us $0 + 0x + 0x^2 = (1 + x + x^2) - (x + x^2 + x^3) - (1 - x^3)$, which means that $1 - x^3 = (1 + x + x^2) - (x + x^2 + x^3)$. And this means that $1 - x^3$ is a dependent member of $\mathcal{B}$. And so we remove $1 - x^3$, leaving us with $\mathcal{B}_1 = \{1 + x + x^2, x + x^2 + x^3\}$. By Theorem A we know that $\mathcal{B}_1$ is a spanning set for Span $\mathcal{B}$. Now let's check to see if $\mathcal{B}_1$ is linearly independent. To that end, we will look for solutions to the equation

$$0+0x+0x^2 = t_1(1+x+x^2)+t_2(x+x^2+x^3) = (t_1)+(t_1+t_2)x+(t_1+t_2)x^2+(t_2)x^3$$

Setting the coefficients equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{aligned} t_1 &= 0 \\ t_1 + t_2 &= 0 \\ t_1 + t_2 &= 0 \\ t_2 &= 0 \end{aligned}$$

The first equation tells us the $t_1 = 0$, and the last equation tells us that $t_2 = 0$, so we see that the only solution to this system is $t_1 = t_2 = 0$. This means that $\mathcal{B}_1$ is linearly independent, and thus $\mathcal{B}_1 = \{1 + x + x^2, x + x^2 + x^3\}$ is a basis for Span $\mathcal{B}$.

B6(b) Since we already know that $\mathcal{B}$ is a spanning set for Span $\mathcal{B}$, we want to check if $\mathcal{B}$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{aligned} 0 + 0x + 0x^2 &= t_1(1 + x + x^2) + t_2(x + x^2 + x^3) + t_3(1 + x^2 + x^3) \\ &= (t_1 + t_3) + (t_1 + t_2)x + (t_1 + t_2 + t_3)x^2 + (t_2 + t_3)x^3 \end{aligned}$$

Setting the coefficients equal, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{cccc} t_1 & & +t_3 & = 0 \\ t_1 & +t_2 & & = 0 \\ t_1 & +t_2 & +t_3 & = 0 \\ & t_2 & +t_3 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 2R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 3. Since this is equal to the number of variables, we know that our equation has a unique solution, which means that $\mathcal{B}$ is linearly dependent. This means that $\mathcal{B}$ is itself a basis for Span $\mathcal{B}$.