Solution to Practice 1f

B1(a) Let $A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_1 + x_3 = x_4,\ x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$.

S0: A is defined as a subset of $\mathbb{R}^4$, and since 0+0=0, we have $\vec{0} \in A$, so A is non-empty.

S1: Let $\vec{x}, \vec{y} \in A$, and let $\vec{z} = \vec{x} + \vec{y}$. Then

$$\begin{aligned} z_1 + z_3 &= (x_1 + y_1) + (x_3 + y_3) && \text{definition of } \vec{z} \\ &= (x_1 + x_3) + (y_1 + y_3) && \text{properties of } \mathbb{R} \\ &= x_4 + y_4 && \text{because } \vec{x}, \vec{y} \in A \\ &= z_4 && \text{definition of } \vec{z} \end{aligned}$$

As we have shown that $z_1 + z_3 = z_4$, we have that $\vec{z} \in A$, and thus A is closed under addition.

S2: Let $\vec{x} \in A$, $t \in \mathbb{R}$, and let $\vec{w} = t\vec{x}$. Then

$$\begin{aligned} w_1 + w_3 &= tx_1 + tx_3 && \text{definition of } \vec{w} \\ &= t(x_1 + x_3) && \text{properties of } \mathbb{R} \\ &= tx_4 && \text{because } \vec{x} \in A \\ &= w_4 && \text{definition of } \vec{w} \end{aligned}$$

As we have shown that $w_1 + w_3 = w_4$, we have that $\vec{w} \in A$, and thus A is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, A is a subspace of $\mathbb{R}^4$.

B1(b) Let $B = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_1 + a_3 = a_4,\ a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$.

S0: B is defined as a subset of $M(2,2)$, and since 0+0=0, we see that the zero matrix is in B, so B is non-empty.

S1: Let $X, Y \in B$, and let $Z = X + Y$. Then

$$\begin{aligned} z_1 + z_3 &= (x_1 + y_1) + (x_3 + y_3) && \text{definition of } Z \\ &= (x_1 + x_3) + (y_1 + y_3) && \text{properties of } \mathbb{R} \\ &= x_4 + y_4 && \text{because } X, Y \in B \\ &= z_4 && \text{definition of } Z \end{aligned}$$

As we have shown that $z_1 + z_3 = z_4$, we have that $Z \in B$, and thus B is closed under addition.

S2: Let $X \in B$, $t \in \mathbb{R}$, and let $W = tX$. Then

$$\begin{aligned}
w_1 + w_3 &= tx_1 + tx_3 && \text{definition of } W \\
&= t(x_1 + x_3) && \text{properties of } \mathbb{R} \\
&= tx_4 && \text{because } X \in B \\
&= w_4 && \text{definition of } W
\end{aligned}$$

As we have shown that $w_1 + w_3 = w_4$, we have that $W \in B$, and thus B is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, B is a subspace of $M(2,2)$.

B1(c) Let $C = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + a_2 = a_3,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$.

S0: C is defined as a subset of P_3, and since 0+0=0, we see that the zero polynomial is in C, so C is non-empty.

S1: Let $a(x), b(x) \in C$, and let $c(x) = a(x) + b(x)$. Then

$$\begin{aligned}
c_0 + c_2 &= (a_0 + b_0) + (a_2 + b_2) && \text{definition of } c(x) \\
&= (a_0 + a_2) + (b_0 + b_2) && \text{properties of } \mathbb{R} \\
&= a_3 + b_3 && \text{because } a(x), b(x) \in C \\
&= c_3 && \text{definition of } c(x)
\end{aligned}$$

As we have shown that $c_0 + c_2 = c_3$, we have that $c(x) \in C$, and thus C is closed under addition.

S2: Let $a(x) \in C$, $t \in \mathbb{R}$, and let $d(x) = ta(x)$. Then

$$\begin{aligned}
d_0 + d_2 &= ta_0 + ta_2 && \text{definition of } d(x) \\
&= t(a_0 + a_2) && \text{properties of } \mathbb{R} \\
&= ta_3 && \text{because } a(x) \in C \\
&= d_3 && \text{definition of } d(x)
\end{aligned}$$

As we have shown that $d_0 + d_2 = d_3$, we have that $d(x) \in C$, and thus C is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, C is a subspace of P_3.

B1(d) Let $D = \{a_0 + a_1x^2 \mid a_0, a_1 \in \mathbb{R}\}$

S0: D is defined as a subset of P_2, but this is also a subset of P_4. And since $0 + 0x^2 = 0 + 0x + 0x^2 + 0x^3 + 0x^4$, the zero polynomial is in D, so D is non-empty.

S1: Let $a(x), b(x) \in D$. Then $a(x) + b(x) = (a_0 + a_1x^2) + (b_0 + b_1x^2) = (a_0 + b_0) + (a_1 + b_1)x^2$. So, if we let $c_0 = a_0 + b_0$, and $c_1 = a_1 + b_1$, then we have that $a(x) + b(x) = c_0 + c_1x^2$, where $c_0, c_1 \in \mathbb{R}$, and thus $a(x) + b(x) \in D$. So D is closed under addition.

S2: Let $a(x) \in D$ and $t \in \mathbb{R}$. Then $ta(x) = t(a_0 + a_1x^2) = (ta_0) + (ta_1)x^2$.

So, if we let $d_0 = ta_0$ and $d_1 = ta_1$, then we see that $ta(x) = d_0 + d_1x^2$ where $d_0, d_1 \in \mathbb{R}$, and thus $ta(x) \in D$. So D is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, D is a subspace of P_4. (And P_3, and P_2.)

B1(e) Let $E = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$ Then the zero matrix is not in E, so E cannot be a vector space, and thus cannot be a subspace of $M(2,2)$.

B1(f) Let $F = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \mid a_1 - a_3 = 1,\ a_1, a_2, a_3 \in \mathbb{R} \right\}$. Then the zero matrix is not in F, since 0-0=0, not 1. So F cannot be a vector space, and thus cannot be a subspace of $M(2,2)$.

B2(a) Let $\mathcal{A} = \{A \in M(3,3) \mid \text{tr}(A) = 0\}$.

S0: $\mathcal{A}$ is defined as a subset of $M(3,3)$, and since $\text{tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = 0 + 0 + 0 = 0$, we see that the zero matrix is in $\mathcal{A}$, and thus that $\mathcal{A}$ is non-empty.

S1: Let $A, B \in \mathcal{A}$, and let $C = A + B$. Then

$$\begin{aligned} \text{tr}(C) &= c_{11} + c_{22} + c_{33} \\ &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + (a_{33} + b_{33}) \\ &= (a_{11} + a_{22} + a_{33}) + (b_{11} + b_{22} + b_{33}) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

As we have shown that $\text{tr}(C) = 0$, we see that $C = A + B \in \mathcal{A}$, and thus $\mathcal{A}$ is closed under addition.

S2: Let $A \in \mathcal{A}$, $t \in \mathbb{R}$, and let $D = tA$. Then

$$\begin{aligned} \text{tr}(D) &= d_{11} + d_{22} + d_{33} \\ &= ta_{11} + ta_{22} + ta_{33} \\ &= t(a_{11} + a_{22} + a_{33}) \\ &= t(0) \\ &= 0 \end{aligned}$$

As we have shown that $\text{tr}(D) = 0$, we see that $D = tA \in \mathcal{A}$, and thus $\mathcal{A}$ is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, $\mathcal{A}$ is a subspace of $M(3,3)$.

B2(b) The subset of invertible matrices is not a subspace of $M(3,3)$ because is does not contain the zero matrix.

B2(c) Let $\mathcal{C} = \left\{ A \in M(3,3) \mid A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

S0: $\mathcal{C}$ is defined as a subset of $M(3,3)$, and since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we see that the zero matrix is in $\mathcal{C}$, and thus that $\mathcal{C}$ is non-empty.

S1: Let $A, B \in \mathcal{C}$. Then $(A+B) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + B \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $(A+B) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we see that $A + B \in \mathcal{C}$, and thus $\mathcal{C}$ is closed under addition.

S2: Let $A \in \mathcal{C}$, $t \in \mathbb{R}$. Then $(tA) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = t \left(A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = t \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $(tA) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we see that $tA \in \mathcal{C}$, and thus $\mathcal{C}$ is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, $\mathcal{C}$ is a subspace of $M(3,3)$.

B2(d) Let $\mathcal{D} = \left\{ A \in M(3,3) \mid A \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$. Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, we see that the zero matrix is not in $\mathcal{D}$, and thus $\mathcal{D}$ is not a subspace of $M(3,3)$.

B2(e) Let $\mathcal{E} = \{A \in M(3,3) \mid A^T = -A\}$. Note that $A^T = -A$ if and only if $a_{ij} = -a_{ji}$ for all $1 \leq i, j \leq 3$.

S0: $\mathcal{E}$ is defined as a subset of $M(3,3)$, and since 0=-0, we see that the zero matrix is in $\mathcal{E}$, and thus that $\mathcal{E}$ is non-empty.

S1: Let $A, B \in \mathcal{E}$, and let $C = (A+B)$. Then for all i and j we see that

$$\begin{aligned} c_{ij} &= a_{ij} + b_{ij} \\ &= -a_{ji} - b_{ji} \\ &= -(a_{ji} + b_{ji}) \\ &= -c_{ji} \end{aligned}$$

As such, $C^T = -C$, so $C = (A + B) \in \mathcal{E}$, and thus $\mathcal{E}$ is closed under addition.

S2: Let $A \in \mathcal{E}$, $t \in \mathbb{R}$, and let $D = tA$. Then for all i and j we see that

$$\begin{aligned} d_{ij} &= ta_{ij} \\ &= t(-a_{ji}) \\ &= -(ta_{ji}) \\ &= -d_{ji} \end{aligned}$$

As such, $D^T = -D$, so $D = tA \in \mathcal{E}$, and thus $\mathcal{E}$ is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, $\mathcal{E}$ is a subspace of $M(3, 3)$.

B3(a) Let $A = \{p(x) \in P_5 \mid p(-x) = p(x) \text{ for all } x \in \mathbb{R}\}$.

S0: A is defined as a subset of P_5, and since $\mathbf{0}(-x) = 0 = \mathbf{0}(x)$ for all $x \in \mathbb{R}$, we see that the zero polynomial is in A, and thus A is non-empty.

S1: Let $p(x), q(x) \in A$. Then $(p + q)(-x) = p(-x) + q(-x) = -p(x) - q(x) = -(p(x) + q(x)) = -(p + q)(x)$. And since $(p + q)(-x) = -(p + q)(x)$, we have that $(p + q)(x) \in A$, and thus A is closed under addition.

S2: Let $p(x) \in A$ and $t \in \mathbb{R}$. Then $(tp)(-x) = t(p(-x)) = t(-p(x)) = -(t(p(x))) = -(tp)(x)$. And since $(tp)(-x) = -(tp)(x)$, we have that $(tp)(x) \in A$, and thus A is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, A is a subspace of P_5.

B3(b) Let $B = \{(p(x))^2 \mid p(x) \in P_2\}$. Then $x^2 \in B$ (since $x^2 = (x)^2$, where $x \in P_2$), and $x^4 \in B$ (since $x^4 = (x^2)^2$, where $x^2 \in P_2$). But $x^2 + x^4 \notin B$, since $x^2 + x^4 = x^2(1 + x^2) \neq (p(x))^2$ for any $p(x) \in P_2$. As such, B is not closed under addition, and therefore is not a subspace of P_5.

B3(c) Let $C = \{a_0 + a_1x + \cdots + a_4x^4 \mid a_1a_4 = 1,\ a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$. Then the zero polynomial is not in C, since (0)(0)=0, not 1. As such, C is not a subspace of P_5.

B3(d) Let $D = \{x^3p(x) \mid p(x) \in P_2\}$.

S0: D is defined as a subset of P_5, and since $\mathbf{0}(x) \in P_2$ and $x^3(\mathbf{0}(x)) = \mathbf{0}(x)$, we see that the zero polynomial is in D, and thus D is non-empty.

S1: Let $p(x), q(x) \in D$, with $p(x) = x^3 a(x)$ and $q(x) = x^3 b(x)$ $(a(x), b(x) \in P_2)$. Then $(p+q)(x) = x^3 a(x) + x^3 b(x) = x^3(a(x) + b(x)) = x^3((a+b)(x))$. Since $a(x), b(x) \in P_2$ (and since P_2 is closed under addition), we have that $(a+b)(x) \in P_2$. Therefore, we have that $(p+q)(x) = x^3(a+b)(x)$ for $(a+b)(x) \in P_2$, which means that $(p+q)(x) \in D$. And thus, D is closed under addition.

S2: Let $p(x) \in D$ (with $p(x) = x^3 a(x)$ for $a(x) \in P_2$), and $t \in \mathbb{R}$. Then $(tp)(x) = t(p(x)) = t(x^3 a(x)) = x^3(ta)(x)$. Because P_2 is a vector space, we know it is closed under scalar multiplication, which means that $(ta)(x) \in P_2$. Therefore, we have that $(tp)(x) = x^3(ta)(x)$ for $(ta)(x) \in P_2$, which means that $(tp)(x) \in D$. And thus, D is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, D is a subspace of P_5.

B3(e) Let $E = \{p(x) \in P_5 \mid p(1) = 0\}$.

S0: E is defined as a subset of P_5, and since $\mathbf{0}(1) = 0$, we see that the zero polynomial is in E, and thus E is non-empty.

S1: Let $p(x), q(x) \in E$. Then $(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$. So $(p+q)(1) = 0$, which means that $(p+q)(x) \in E$. And so we see that E is closed under addition.

S2: Let $p(x) \in E$ and $t \in \mathbb{R}$. Then $(tp)(1) = t(p(1)) = t(0) = 0$. So $(tp)(1) = 0$, which means that $(tp)(x) \in E$. And so we see that E is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, E is a subspace of P_5.

B4(a) Let $A = \{f \in \mathcal{F} \mid f(3) + f(5) = 0\}$.

S0: First we note that A is defined to be a subset of $\mathcal{F}$. Moreover, since $\mathbf{0}(3) + \mathbf{0}(5) = 0 + 0 = 0$, we see that the zero function is in A, and thus A is non-empty.

S1: Let $f, g \in A$. Then $(f+g)(3) + (f+g)(5) = (f(3) + g(3)) + (f(5) + g(5)) = (f(3) + f(5)) + (g(3) + g(5)) = 0 + 0 = 0$. And since $(f+g)(3) + (f+g)(5) = 0$, we have that $f + g \in A$, which shows that A is closed under addition.

S2: Let $f \in A$ and $t \in \mathbb{R}$. Then $(tf)(3) + (tf)(5) = t(f(3)) + t(f(5)) = t(f(3) + f(5)) = t(0) = 0$. And since $(tf)(3) + (tf)(5) = 0$, we have that $tf \in A$, which shows that A is closed under scalar multiplication.

Since properties S0, S1, and S2 all hold, A is a subspace of $\mathcal{F}$.

B4(b) Let $B = \{f \in \mathcal{F} \mid f(1) + f(2) = 1\}$. Then the zero function is not in B, since $\mathbf{0}(1) + \mathbf{0}(2) = 0 + 0 = 0 \neq 1$. As such, B is not a vector space, and thus cannot be a subspace of $\mathcal{F}$.

B4(c) Let $C = \{f \in \mathcal{F} \mid |f(x)| \leq 1\}$. Then the constant function $f(x) = 1$ is in C, but the scalar multiple $2f(x)$ is not in C, since $2f(x) = 2$ for all x, which means that $|2f(x)| > 1$, and thus $2f \notin C$. Since C is not closed under scalar multiplication, C is not a subspace of $\mathcal{F}$.

B4(d) Let $D = \{f \in \mathcal{F} \mid f \text{ is increasing on } \mathbb{R}\}$. That is, $f \in D$ if and only if we have $f(x) \leq f(y)$ whenever $x \leq y$.

Then D is not a subspace of $\mathcal{F}$, because D is not closed under scalar multiplication. For a counterexample, consider that the function $f(x) = x$ is in D, but the scalar multiple $-f(x)$ is a decreasing function, and thus not in D.