

Lecture 1k  
Extending a Linearly Independent Subset to a Basis  
(pages 213-216)

Now that we know that the vector spaces in this course have a finite number of vectors in their basis, we can proceed to extend any linearly independent subset to a basis. And the way we do so is easy—just pick a vector not already in the span, and add it.

Theorem 4.3.b: If  $\mathcal{T} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a linearly independent set, and if  $\mathbf{w} \notin \text{Span } \mathcal{T}$ , then  $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}\}$  is also a linearly independent set.

Proof of Theorem 4.3.b: Suppose that  $\mathcal{T} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a linearly independent set, and that  $\mathbf{w} \notin \text{Span } \mathcal{T}$ . To see that  $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}\}$  is linearly independent, we will look for solutions to the equation

$$t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k + t_{k+1}\mathbf{w} = \mathbf{0}$$

Suppose, by way of contradiction, that  $t_{k+1} \neq 0$ . Then we can divide by  $t_{k+1}$  and get

$$\frac{t_1}{t_{k+1}}\mathbf{v}_1 + \dots + \frac{t_k}{t_{k+1}}\mathbf{v}_k + \mathbf{w} = \mathbf{0}$$

which means that

$$-\frac{t_1}{t_{k+1}}\mathbf{v}_1 - \dots - \frac{t_k}{t_{k+1}}\mathbf{v}_k = \mathbf{w}$$

But this means we can write  $\mathbf{w}$  as a linear combination of the vectors in  $\mathcal{T}$ , which contradicts our choice of  $\mathbf{w}$  as not being in the span of  $\mathcal{T}$ . From this contradiction, we know that  $t_{k+1} = 0$ . And this turns our linear independence equation into

$$t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k + 0\mathbf{w} = \mathbf{0}$$

which is the same as

$$t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k = \mathbf{0}$$

And since  $\mathcal{T}$  is linearly independent, we know that the only solution to this equation is  $t_1 = \dots = t_k = 0$ . As such, we have shown that the only solution to the equation

$$t_1 \mathbf{v}_1 + \cdots + t_k \mathbf{v}_k + t_{k+1} \mathbf{w} = \mathbf{0}$$

is  $t_1 = \cdots = t_k = t_{k+1} = 0$ . And this means that the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}\}$  is linearly independent.

Now, the fact that our vector spaces have a finite basis does more than guarantee that our expansion process will come to an end. It actually tells us *when* our process will end. Because our basis will need to have exactly  $\dim \mathbb{V}$  elements in it. We see how to use this fact in the following example.

**Example:** (a) Produce a basis  $\mathcal{B}$  for the plane  $\mathcal{P}$  in  $\mathbb{R}^3$  with equation  $2x_1 + 4x_2 - x_3 = 0$ , and (b) extend the basis  $\mathcal{B}$  to a basis  $\mathcal{C}$  for  $\mathbb{R}^3$ .

We already know that the dimension of any plane in  $\mathbb{R}^3$  is 2, so to find a basis  $\mathcal{B}$  for  $\mathcal{P}$ , we simply need to find two linearly independent vectors on our plane. But a set of two vectors is linearly independent whenever they are not

a scalar multiple of each other. So we can quickly note that  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$  are two vectors that satisfy  $2x_1 + 4x_2 - x_3 = 0$  that are not scalar

multiples of each other. And thus,  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$  is a basis for  $\mathcal{P}$ .

Now, we know that the dimension of  $\mathbb{R}^3$  is 3, so to extend  $\mathcal{B}$  to a basis  $\mathcal{C}$  of  $\mathbb{R}^3$ , we simply need to find one vector not in the span of  $\mathcal{B}$ . But the span of  $\mathcal{B}$  is precisely the vectors on the plane  $\mathcal{P}$ . So this means we are looking for any vector not on the plane. That is, we are looking for any vector that does NOT

satisfy  $2x_1 + 4x_2 - x_3 = 0$ . The vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  quickly comes to mind as such a vector, and so we have that  $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^3$ .

During the process of shrinking spanning sets and expanding linearly independent sets, we have discovered the following:

**Theorem 4.3.4** Let  $\mathbb{V}$  be an  $n$ -dimensional vector space. Then

- (1) A set of more than  $n$  vectors in  $\mathbb{V}$  must be linearly dependent.
- (2) A set of fewer than  $n$  vectors cannot span  $\mathbb{V}$ .
- (3) A set with  $n$  elements of  $\mathbb{V}$  is a spanning set for  $\mathbb{V}$  if and only if it is linearly

independent.

Course Author's note: I refer to part (3) as the “two-out-of three” rule for proving that something is a basis. By definition, a basis must be a spanning set and linearly independent. But we also know that our basis must have the correct number of elements. So, really there are three features that our basis must have: spanning, linear independence, and  $n$  elements. Between the definition of a basis and part (3), we see that showing any two of these three features will guarantee that our set is a basis.