Lecture 1h

Bases

(pages 206-209)

The next step after defining a spanning set and linear independence is to look at a basis–a set that is both at the same time! We did not pay much attention to bases in Math 106, but they will play a much greater role in this class, and the following theorem is the reason why.

Theorem 4.3.1 **(Unique Representation Theorem)**: Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for a vector space $\mathbb{V}$. Then every vector in $\mathbb{V}$ can be expressed in a *unique* way as a linear combination of the vectors of $\mathcal{B}$ if and only if the set $\mathcal{B}$ is linearly independent.

Proof of the Unique Representation Theorem: Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for a vector space $\mathbb{V}$. Our "if and only if" proof consists of two parts:

Part 1: ($\Rightarrow$) If every vector in $\mathbb{V}$ can be expressed as a unique linear combination of the vectors in $\mathcal{B}$, then $\mathcal{B}$ is linearly independent.

To see this, we note that if *every* vector in $\mathbb{V}$ can be expressed as a unique linear combination of the vectors in $\mathcal{B}$, then we specifically know that the zero vector can be expressed as a unique linear combination of the vectors in $\mathcal{B}$. This means that we know there is only one collection of scalars $t_1, \ldots, t_n$ such that

$$t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n = \mathbf{0}$$

And since we know that $t_1 = \ldots = t_n = 0$ is such a collection, we have that this is the only collection, and so, by definition, $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

Part 2: ($\Leftarrow$) If $\mathcal{B}$ is linearly independent, then every vector in $\mathbb{V}$ can be expressed as a unique linear combination of the vectors in $\mathcal{B}$.

Like all "unique" theorems, we will assume that we have two expressions for some $\mathbf{x}$, and show that they are in fact the same. To that end, let $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}$ be such that

$$\begin{aligned} \mathbf{x} &= a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \quad \text{and} \\ \mathbf{x} &= b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n \end{aligned}$$

From this we see that

$$\begin{aligned} \mathbf{0} &= \mathbf{x} - \mathbf{x} \\ &= (a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) - (b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n) \\ &= (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n \end{aligned}$$

Since $\mathcal{B}$ is linearly independent, then the only solution to the equation

$$\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n$$

is the trivial solution $a_i - b_i = 0$ for all $1 \leq i \leq n$. This means that $a_i = b_i$ for all $1 \leq i \leq n$, as so we see that if $\mathcal{B}$ is linearly independent, then $a_i = b_i$ for all $1 \leq i \leq n$. Which means that there is only one way to write $\mathbf{x}$ as a linear combination of the vectors in $\mathcal{B}$.

Definition: A set $\mathcal{B}$ of vectors in a vector space $\mathbb{V}$ is a **basis** for $\mathbb{V}$ if it is a linearly independent spanning set for $\mathbb{V}$.

Note: Using this definition, the vector space $\mathbb{O} = \{\mathbf{0}\}$ cannot have a basis, since the only set of vectors from $\mathbb{O}$ is $\mathbb{O}$ itself. But since the set $\{\mathbf{0}\}$ contains the zero vector, it is not linearly independent. However, we would like every vector space to have a basis. Therefore, we define the basis of $\{\mathbf{0}\}$ to be the empty set.

The Unique Representation Theorem will be a very powerful tool for us to use, but in order to use it we will need to have a basis for our vector space. The first step in this process will be making sure that we can identify a basis.

Example: Show that $\mathcal{A} = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} \right\}$ is a basis for $M(2,2)$.

To show that $\mathcal{A}$ is a basis for $M(2,2)$, we need to show that it is a spanning set for $M(2,2)$ and that it is linearly independent. To see that it is a spanning set for $M(2,2)$, we need to see that there is a solution to the equation

$$t_1 \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + t_2 \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} + t_3 \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix} + t_4 \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for any choice of $a, b, c, d \in \mathbb{R}$. Performing the calculation on the left, we get

$$\begin{bmatrix} t_1 - t_2 + 3t_3 - t_4 & 2t_1 + 2t_3 + 4t_4 \\ 3t_3 - t_2 + 8t_3 + t_4 & t_4 + 2t_2 - 3t_3 + 7t_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the following system:

$$\begin{array}{rrrrl} t_1 & -t_2 & +3t_3 & -t_4 & = a \\ 2t_1 & & +2t_3 & +4t_4 & = b \\ 3t_1 & -t_2 & +8t_3 & +t_4 & = c \\ t_1 & +2t_2 & -3t_3 & +7t_4 & = d \end{array}$$

To see if this system has solutions, we need to row reduce its augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & -1 & a \\ 2 & 0 & 2 & 4 & b \\ 3 & -1 & 8 & 1 & c \\ 1 & 2 & -3 & 7 & d \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & -1 & a \\ 0 & 2 & -4 & 6 & b-2a \\ 0 & 2 & -1 & 4 & c-3a \\ 0 & 3 & -6 & 8 & d-a \end{array}\right] \begin{array}{l} \\ (1/2)R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & -1 & a \\ 0 & 1 & -2 & 3 & (1/2)b-a \\ 0 & 2 & -1 & 4 & c-3a \\ 0 & 3 & -6 & 8 & d-a \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & -1 & a \\ 0 & 1 & -2 & 3 & (1/2)b-a \\ 0 & 0 & 3 & -2 & c-b-a \\ 0 & 0 & 0 & -1 & d-(3/2)b+2a \end{array}\right]$$

This last matrix is in row echelon form, and since it does not have any bad rows, we see that our system does have a solution. Which means that $\mathcal{A}$ is a spanning set for $M(2,2)$. So, now we want to verify that $\mathcal{A}$ is linearly independent. To do this, we need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + t_2 \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} + t_3 \begin{bmatrix} 3 & 2 \\ 8 & -3 \end{bmatrix} + t_4 \begin{bmatrix} -1 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

As this is just a special case of our earlier equation (with $a = b = c = d = 0$), we see that this is equivalent to the system

$$\begin{array}{rrrrl} t_1 & -t_2 & +3t_3 & -t_4 & = 0 \\ 2t_1 & & +2t_3 & +4t_4 & = 0 \\ 3t_1 & -t_2 & +8t_3 & +t_4 & = 0 \\ t_1 & +2t_2 & -3t_3 & +7t_4 & = 0 \end{array}$$

To find the solutions of this homogeneous system, we row reduce the coefficient matrix. We use the same steps as before to see that a row echelon form of the coefficient matrix is the following:

$$\begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

From this, we see that the rank of the coefficient matrix is 4, which is the same as the number of columns, so the system has a unique solution. And from this, we know that $\mathcal{A}$ is linearly independent. And, having shown that $\mathcal{A}$ is a linearly independent spanning set for $M(2,2)$, we have shown that $\mathcal{A}$ is a basis for $M(2,2)$.

Example: Show that $\mathcal{B} = \{1, 1+x, 1+x^2, x+x^2\}$ is not a basis for P_2.

If $\mathcal{B}$ is not a basis for P_2, then it either is not a spanning set for P_2, or it is not linearly independent. Let's look at the question of span first. To see if $\mathcal{B}$ is a spanning set for P_2, we need to see if the equation

$$t_1(1) + t_2(1+x) + t_3(1+x^2) + t_4(x+x^2) = p_0 + p_1x + p_2x^2$$

has a solution for every $p_0 + p_1x + p_2x^2 \in P_2$. Doing the calculation on the left, we see we are looking at:

$$(t_1 + t_2 + t_3) + (t_2 + t_4)x + (t_3 + t_4)x^2 = p_0 + p_1x + p_2x^2$$

And setting the coefficients equal to each other, we see that this is equivalent to the following system of linear equations:

$$\begin{array}{ccccc} t_1 & +t_2 & +t_3 & & = p_0 \\ & t_2 & +t_3 & & = p_1 \\ & & t_3 & +t_4 & = p_2 \end{array}$$

To solve this system, we will look at its augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & p_0 \\ 0 & 1 & 0 & 1 & p_1 \\ 0 & 0 & 1 & 1 & p_2 \end{array}\right]$$

This matrix is already in row echelon form, and we see that there are no bad rows. This means that there IS a solution, which means that $\mathcal{B}$ is a spanning set for P_2. So, it must be that $\mathcal{B}$ is linearly dependent. To show this, we need to show that there are non-trivial solutions to the equation

$$t_1(1) + t_2(1+x) + t_3(1+x^2) + t_4(x+x^2) = 0 = 0 + 0x + 0x^2$$

Note that this is simply a specific example of the equation we were looking at before, with $p_0 = p_1 = p_2 = 0$. As such, we know that the solution to this equation can be found by looking at the following matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

This matrix is already in row echelon form, and of interest to us now is that the rank of the coefficient matrix is 3. Since this is less than the number of columns in the coefficient matrix, there are parameters in the general solution, which means that the solution is not unique. Thus, our set is not linearly independent, and therefore is not a basis.

One of the unusual properties of the polynomial spaces is that any given polynomial is a member of not just one space, but instead of all the spaces big enough. So, we do not know just from looking at a polynomial what space we are talking about. (This never happens in $\mathbb{R}^n$ or $M(m,n)$, where we are very specific about the number of entries in each member.) But, these polynomials do have different properties, depending on the space they are in. Consider the following example.

Example: Show that $\mathcal{C} = \{1+x+x^2, 1-x-2x^2, 4x\}$ is a basis for P_2, but not a basis for P_3.

First we want to show that Span $\mathcal{C} = P_2$, and that $\mathcal{C}$ is linearly independent. To show span, we need to see that the equation

$$t_1(1+x+x^2) + t_2(1-x-2x^2) + t_3(4x) = p_0 + p_1x + p_2x^2$$

has a solution for every $p_0 + p_1x + p_2x^2 \in P_2$. Doing the calculation on the left, we see we are looking at:

$$(t_1+t_2) + (t_1 - t_2 + 4t_3)x + (t_1 - 2t_2)x^2 = p_0 + p_1x + p_2x^2$$

And setting the coefficients equal to each other, we see that this is equivalent to the following system of linear equations:

$$\begin{array}{cccl} t_1 & +t_2 & & = p_0 \\ t_1 & -t_2 & +4t_3 & = p_1 \\ t_1 & -2t_2 & & = p_2 \end{array}$$

To solve this system, we will row reduce its augmented matrix:

$$\left[\begin{array}{rrr|l} 1 & 1 & 0 & p_0 \\ 1 & -1 & 4 & p_1 \\ 1 & -2 & 0 & p_2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 1 & 0 & p_0 \\ 0 & -2 & 4 & p_1 - p_0 \\ 0 & -3 & 0 & p_2 - p_0 \end{array}\right] \ (-1/2)R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 1 & 0 & p_0 \\ 0 & 1 & -2 & (-1/2)p_1 + (1/2)p_0 \\ 0 & -3 & 0 & p_2 - p_0 \end{array}\right] R_3 + 3R_2$$

$$\sim \left[\begin{array}{rrr|c} 1 & 1 & 0 & p_0 \\ 0 & 1 & -2 & (-1/2)p_1 + (1/2)p_0 \\ 0 & 0 & -6 & p_2 - (3/2)p_1 + (1/2)p_0 \end{array}\right]$$

Our augmented matrix is now in row echelon form. And since there are no bad rows, we know that our system has a solution, and thus that $\mathcal{C}$ is a spanning set for P_2. Now we need to show that $\mathcal{C}$ is linearly independent. That means we need to look for solutions to the equation

$$t_1(1 + x + x^2) + t_2(1 - x - 2x^2) + t_3(4x) = 0 = 0 + 0x + 0x^2$$

This is just a special case of the equation we were looking at, so we can simply plug in $p_0 = p_1 = p_2 = 0$ into our previous work, and we will end up looking at the following row reduced coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -6 \end{bmatrix}$$

Since the rank of this matrix is 3, which equals the number of columns in our coefficient matrix, our system has a unique solution. And this means that our set is linearly independent. As such, we have seen that $\mathcal{C}$ is a basis for P_2.

But what changes when we consider $\mathcal{C}$ as a subset of P_3? Well, we need to add a "$+0x^3$" to every polynomial, and a corresponding "$+p_3x^3$" to our general polynomial. So, when we consider whether or not $\mathcal{C}$ is a spanning set for P_3, we are now looking for solutions to:

$$t_1(1+x+x^2+0x^3)+t_2(1-x-2x^2+0x^3)+t_3(4x+0x^3) = p_0+p_1x+p_2x^2+p_3x^3$$

This turns our system of equations into the following:

$$\begin{array}{rrrl} t_1 & +t_2 & & = p_0 \\ t_1 & -t_2 & +4t_3 & = p_1 \\ t_1 & -2t_2 & & = p_2 \\ 0 & 0 & 0 & = p_3 \end{array}$$

(Normally we wouldn't bother adding all those zero entries in the last row, but I've put them in for emphasis.) We don't even need to put this into matrix form to see that the last row will give us problems. And so, we see that $\mathcal{C}$ is not a basis for P_3, because it is not a spanning set for P_3.

But what about linear independence? We don't need to check this, since we have already shown that $\mathcal{C}$ is not a basis for P_3, but it is interesting to note that $\mathcal{C}$ is still linearly independent in P_3. To see this, note that we are simply looking for solutions in the case when $p_0 = p_1 = p_2 = p_3 = 0$. So we are looking for solutions to the system

$$\begin{array}{cccc} t_1 & +t_2 & & = 0 \\ t_1 & -t_2 & +4t_3 & = 0 \\ t_1 & -2t_2 & & = 0 \\ 0 & 0 & 0 & = 0 \end{array}$$

By setting $p_3 = 0$, we no longer have any problems with our last row. And to find solutions to this system we simply row reduce the coeffient matrix. Using the same steps we did before, we now end up with this matrix in row echelon form:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

The extra row of zeros does not change the fact that the rank of this matrix is 3, which is still the same as the number of COLUMNS, and so there is still a unique solution to our equations. And this means that $\mathcal{C}$ is linearly independent in P_3 as well.

Example: Each of our common vector spaces ($\mathbb{R}^n$, $M(m,n)$, and P_n) have a well known basis, which we refer to as the "standard basis".

The standard basis for $\mathbb{R}^n = \{\vec{e}_1, \ldots, \vec{e}_n\}$, where $\vec{e}_i$ is the vector with a "1" in the i-th entry, and a 0 in all other entries.

Let $E_{i,j}$ be an $m \times n$ matrix with a "1" in the (i,j)-th entry, and a 0 in all other entries. Then the set $\{E_{i,j} \mid 1 \le i \le m, 1 \le j \le n\}$ is the standard basis for $M(m,n)$. For example, the set

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

is the standard basis for $M(2,3)$.

The standard basis for P_n is the set $\{1, x, \ldots, x^n\}$.