

Lecture 1g
Span and Linear Independence in Vector Spaces
(page 204)

One of the common ways to define a subspace is to think of it as the set of all linear combinations of a set of vectors. First, let's note that this IS a subspace!

Theorem 4.2.2 If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space \mathbb{V} , and \mathbb{S} is the set of all possible linear combinations of these vectors,

$$\mathbb{S} = \{t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k \mid t_1, \dots, t_k \in \mathbb{R}\}$$

then \mathbb{S} is a subspace of \mathbb{V} .

Proof of Theorem 4.2.2: Let's look at our three properties:

S0: Since \mathbb{V} is closed under addition and scalar multiplication, we know that every $t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k$ is an element of \mathbb{V} , and thus \mathbb{S} is a subset of \mathbb{V} . And \mathbb{S} is not empty since, at the least, $\mathbf{v}_1 \in \mathbb{S}$.

S1: Let $\mathbf{x} = s_1\mathbf{v}_1 + \dots + s_k\mathbf{v}_k$ and $\mathbf{y} = t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k$ be elements of \mathbb{S} . Then

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (s_1\mathbf{v}_1 + \dots + s_k\mathbf{v}_k) + (t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k) \\ &= s_1\mathbf{v}_1 + t_1\mathbf{v}_1 + \dots + s_k\mathbf{v}_k + t_k\mathbf{v}_k && \text{by V2 and V5} \\ &= (s_1 + t_1)\mathbf{v}_1 + \dots + (s_k + t_k)\mathbf{v}_k && \text{by V8} \end{aligned}$$

And so we see that $\mathbf{x} + \mathbf{y} \in \mathbb{S}$.

S2: Let $\mathbf{x} = s_1\mathbf{v}_1 + \dots + s_k\mathbf{v}_k$ be an element of \mathbb{S} , and let $t \in \mathbb{R}$. Then

$$\begin{aligned} t\mathbf{x} &= t(s_1\mathbf{v}_1 + \dots + s_k\mathbf{v}_k) \\ &= t(s_1\mathbf{v}_1) + \dots + t(s_k\mathbf{v}_k) && \text{by V9} \\ &= (ts_1)\mathbf{v}_1 + \dots + (ts_k)\mathbf{v}_k && \text{by V7} \end{aligned}$$

And so we see that $t\mathbf{x} \in \mathbb{S}$.

And since properties S0, S1, and S2 hold, \mathbb{S} is a subspace of \mathbb{V} .

Example: The set of all diagonal 2×2 matrices is a vector space, since it is the set of all possible linear combinations of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $M(2, 2)$.

As we did with \mathbb{R}^n and $M(m, n)$, we will call the set of all linear combinations the span.

Definition: If \mathbb{S} is the subspace of the vector space \mathbb{V} consisting of all linear combinations of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{V}$, then \mathbb{S} is called the subspace **spanned** by $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, and we say that the set \mathcal{B} **spans** \mathbb{S} . The set \mathcal{B} is called a **spanning set** for the subspace \mathbb{S} . We denote \mathbb{S} by

$$\mathbb{S} = \text{Span } \{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span } \mathcal{B}$$

We also extend our definition of linear dependence and linear independence to general vector spaces.

Definition: If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space \mathbb{V} , then \mathcal{B} is said to be linearly independent if the only solution to the equation

$$t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k = \mathbf{0}$$

is $t_1 = \dots = t_k = 0$; otherwise, \mathcal{B} is said to be **linearly dependent**.

I won't go over examples of spanning sets and linear independence, since we've already looked at these extensively in our studies of \mathbb{R}^n , $M(m, n)$, and P_n . But we do have an opportunity now to prove a statement about linear independence in ALL vector spaces.

Theorem 4.2.a: Any set that contains the zero vector is linearly dependent.

Note: From time to time, I will add theorems that are not in the text. Or, more often, statements that are simply not stated as theorems in the text. I will "number" these theorems with letters, so that they do not interfere with the numbering system in the textbook.

Proof: Let \mathbb{V} be a vector space, and let $\mathcal{A} = \{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ be a set of vectors from \mathbb{V} that contains the zero vector. To see that \mathcal{A} is linearly dependent, we need to find a non-trivial solution to the equation

$$t_0\mathbf{0} + t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \dots + t_k\mathbf{x}_k = \mathbf{0}$$

I claim that setting $t_0 = 1$, and $t_1 = t_2 = \dots = t_k = 0$ is such a solution. First, we note that the scalar multiplicative identity property (V10) tells us that $1\mathbf{0} = \mathbf{0}$, so setting $t_0 = 1$ means we can replace $t_0\mathbf{0}$ with $\mathbf{0}$. Next we note that, by Theorem 4.2.1, $0\mathbf{x}_i = \mathbf{0}$ for all $1 \leq i \leq k$, so setting $t_1 = t_2 = \dots = t_k = 0$ means we can replace all the $t_i\mathbf{x}_i$ with $\mathbf{0}$. And so, our equation becomes

$$\mathbf{0} + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$$

which is true thanks to repeated uses of the additive identity property (V3).