Solution to Practice 3x

B1(a) A is not Hermitian

B1(b) B is Hermitian. Let's find the eigenvalues of B:

$$\begin{aligned}
\det\,(B-\lambda I) &= \det \begin{bmatrix} 5-\lambda & \sqrt{2}-i \\ \sqrt{2}+i & 3-\lambda \end{bmatrix} \\
&= (5-\lambda)(3-\lambda) - (2-\sqrt{2}i+\sqrt{2}i-i^2) \\
&= 15-5\lambda-3\lambda+\lambda^2-3 \\
&= 12-8\lambda+\lambda^2 \\
&= (6-\lambda)(2-\lambda)
\end{aligned}$$

So, the eigenvalues of B are $\lambda = 2, 6$. Now let's find the eigenspaces for these eigenvectors:

The eigenspace for $\lambda = 2$ is the nullspace of $\begin{bmatrix} 5-\lambda & \sqrt{2}-i \\ \sqrt{2}+i & 3-\lambda \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{2}-i \\ \sqrt{2}+i & 1 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 3 & \sqrt{2}-i \\ 0 & 0 \end{bmatrix}$. So, the eigenvectors for $\lambda = 2$ satisfy $3z_1 + (\sqrt{2}-i)z_2 = 0$, so we have that $z_1 = (-1/3)(\sqrt{2}-i)z_2$. If we replace the variable z_2 with the parameter α, we see that the eigenvectors of $\lambda = 2$ are $\left\{ \begin{bmatrix} (-1/3)(\sqrt{2}-i)\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}$, which we can write as $\operatorname{Span}\left\{ \begin{bmatrix} (\sqrt{2}-i)/\sqrt{12} \\ -3/\sqrt{12} \end{bmatrix} \right\}$. And so have that $\left\{ \begin{bmatrix} (\sqrt{2}-i)/\sqrt{12} \\ -3/\sqrt{12} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace for $\lambda = 2$.

The eigenspace for $\lambda = 6$ is the nullspace of $\begin{bmatrix} 5-\lambda & \sqrt{2}-i \\ \sqrt{2}+i & 3-\lambda \end{bmatrix} = \begin{bmatrix} -1 & \sqrt{2}-i \\ \sqrt{2}+i & -3 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} -1 & \sqrt{2}-i \\ 0 & 0 \end{bmatrix}$. So, the eigenvectors for $\lambda = 6$ satisfy $-z_1 + (\sqrt{2}-i)z_2 = 0$, so we have that $z_1 = (\sqrt{2}-i)z_2$. If we replace the variable z_2 with the parameter α, we see that the eigenvectors of $\lambda = 6$ are $\left\{ \begin{bmatrix} (\sqrt{2}-i)\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}$, which we can write as $\operatorname{Span}\left\{ \begin{bmatrix} (\sqrt{2}-i)/2 \\ 1/2 \end{bmatrix} \right\}$. And so have that $\left\{ \begin{bmatrix} (\sqrt{2}-i)/2 \\ 1/2 \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace for $\lambda = 6$.

And so we have that the matrix $U = \begin{bmatrix} (\sqrt{2}-i)/\sqrt{12} & (\sqrt{2}-i)/2 \\ -3/\sqrt{12} & 1/2 \end{bmatrix}$ is a unitary matrix such that $U^*BU = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$.

B1(c) C is Hermitian. Let's find the eigenvalues of C:

$$
\begin{aligned}
\det\,(C-\lambda I) &= \det \begin{bmatrix} 5-\lambda & \sqrt{3}+i \\ \sqrt{3}-i & 2-\lambda \end{bmatrix} \\
&= (5-\lambda)(2-\lambda) - (3+\sqrt{3}i - \sqrt{3}i - i^2) \\
&= 10 - 5\lambda - 2\lambda + \lambda^2 - 4 \\
&= 6 - 7\lambda + \lambda^2 \\
&= (6-\lambda)(1-\lambda)
\end{aligned}
$$

So, the eigenvalues of C are $\lambda = 1, 6$. Now let's find the eigenspaces for these eigenvectors:

The eigenspace for $\lambda = 1$ is the nullspace of $\begin{bmatrix} 5-\lambda & \sqrt{3}+i \\ \sqrt{3}-i & 2-\lambda \end{bmatrix} = \begin{bmatrix} 4 & \sqrt{3}+i \\ \sqrt{3}-i & 1 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 4 & \sqrt{3}+i \\ 0 & 0 \end{bmatrix}$. So, the eigenvectors for $\lambda = 1$ satisfy $4z_1 + (\sqrt{3}+i)z_2 = 0$, so we have that $z_1 = (-1/4)(\sqrt{3}+i)z_2$. If we replace the variable z_2 with the parameter α, we see that the eigenvectors of $\lambda = 1$ are $\left\{ \begin{bmatrix} (-1/4)(\sqrt{3}+i)\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}$, which we can write as Span$\left\{ \begin{bmatrix} (\sqrt{3}+i)/\sqrt{20} \\ -4/\sqrt{20} \end{bmatrix} \right\}$. And so have that $\left\{ \begin{bmatrix} (\sqrt{3}+i)/\sqrt{20} \\ -4/\sqrt{20} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace for $\lambda = 2$.

The eigenspace for $\lambda = 6$ is the nullspace of $\begin{bmatrix} 5-\lambda & \sqrt{3}+i \\ \sqrt{3}-i & 2-\lambda \end{bmatrix} = \begin{bmatrix} -1 & \sqrt{3}+i \\ \sqrt{3}-i & -4 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} -1 & \sqrt{3}+i \\ 0 & 0 \end{bmatrix}$. So, the eigenvectors for $\lambda = 6$ satisfy $-z_1 + (\sqrt{3}+i)z_2 = 0$, so we have that $z_1 = (\sqrt{3}+i)z_2$. If we replace the variable z_2 with the parameter α, we see that the eigenvectors of $\lambda = 6$ are $\left\{ \begin{bmatrix} (\sqrt{3}+i)\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}$, which we can write as Span$\left\{ \begin{bmatrix} (\sqrt{3}+i)/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$. And so have that $\left\{ \begin{bmatrix} (\sqrt{3}+i)/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace for $\lambda = 6$.

And so we have that the matrix $U = \begin{bmatrix} (\sqrt{3}+i)/\sqrt{20} & (\sqrt{3}+i)/\sqrt{5} \\ -4/\sqrt{20} & 1/\sqrt{5} \end{bmatrix}$ is a unitary matrix such that $U^*CU = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$.

B1(d) F is Hermitian. Let's find the eigenvalues of F:

$$\begin{aligned}
\det\,(F-\lambda I) &= \det \begin{bmatrix} 1-\lambda & i & -i \\ -i & -1-\lambda & i \\ i & -i & -\lambda \end{bmatrix} \\
&= (1-\lambda)\begin{vmatrix} -1-\lambda & i \\ -i & -\lambda \end{vmatrix} - i\begin{vmatrix} -i & i \\ i & -\lambda \end{vmatrix} - i\begin{vmatrix} -i & -1-\lambda \\ i & -i \end{vmatrix} \\
&= (1-\lambda)(\lambda+\lambda^2+i^2) - i(\lambda i - i^2) - i(i^2 + i_\lambda i) \\
&= \lambda + \lambda^2 - 1 - \lambda^2 - \lambda^3 + \lambda + \lambda - i + i + 1 + \lambda \\
&= 4\lambda - \lambda^3 \\
&= \lambda(\lambda-2)(\lambda+2)
\end{aligned}$$

So, the eigenvalues of F are $\lambda = 0, 2, -2$. Now let's find the eigenspaces for these eigenvectors:

The eigenspace for $\lambda = 0$ is the nullspace of $\begin{bmatrix} 1-\lambda & i & -i \\ -i & -1-\lambda & i \\ i & -i & -\lambda \end{bmatrix} = \begin{bmatrix} 1 & i & -i \\ -i & -1 & i \\ i & -i & 0 \end{bmatrix}$.

To find the nullspace, we need to row reduce this matrix:

$$\begin{bmatrix} 1 & i & -i \\ -i & -1 & i \\ i & -i & 0 \end{bmatrix} \begin{matrix} \\ R_2 + iR_1 \\ R_3 - iR_1 \end{matrix} \sim \begin{bmatrix} 1 & i & -i \\ 0 & -2 & 1+i \\ 0 & 1-i & -1 \end{bmatrix} \; (-1/2)R_2$$

$$\sim \begin{bmatrix} 1 & i & -i \\ 0 & 1 & (-1/2)-(i/2) \\ 0 & 1-i & -1 \end{bmatrix} \begin{matrix} R_1 - iR_2 \\ \\ R_3 + (-1+i)R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & (-1/2)-(i/2) \\ 0 & 1 & (-1/2)-(i/2) \\ 0 & 0 & 0 \end{bmatrix}$$

So, the eigenvectors for $\lambda = 0$ satisfy $z_1 - ((1/2)+(1/2)i)z_3 = 0$ and $z_2 - ((1/2)+(1/2)i)z_3 = 0$. If we replace the variable z_3 with the parameter α, we see that the eigenvectors of $\lambda = 0$ are $\left\{ \begin{bmatrix} ((1/2)+(1/2)i)\alpha \\ ((1/2)+(1/2)i)\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}$, which we can write as Span $\left\{ \begin{bmatrix} (1+i)/\sqrt{8} \\ (1+i)/\sqrt{8} \\ 2/\sqrt{8} \end{bmatrix} \right\}$. And so have that $\left\{ \begin{bmatrix} (1+i)/\sqrt{8} \\ (1+i)/\sqrt{8} \\ 2/\sqrt{8} \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace for $\lambda = 0$.

The eigenspace for $\lambda = 2$ is the nullspace of $\begin{bmatrix} 1-\lambda & i & -i \\ -i & -1-\lambda & i \\ i & -i & -\lambda \end{bmatrix} = \begin{bmatrix} -1 & i & -i \\ -i & -3 & i \\ i & -i & -2 \end{bmatrix}$.

To find the nullspace, we need to row reduce this matrix:

$$\begin{bmatrix} -1 & i & -i \\ -i & -3 & i \\ i & -i & -2 \end{bmatrix} \begin{matrix} -R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -i & i \\ -i & -3 & i \\ i & -i & -2 \end{bmatrix} \begin{matrix} \\ R_2 + iR_1 \\ R_3 - iR_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -i & i \\ 0 & -2 & -1+i \\ 0 & -1-i & -1 \end{bmatrix} \begin{matrix} \\ (-1/2)R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -i & i \\ 0 & 1 & (1/2)-(1/2)i \\ 0 & -1-i & -1 \end{bmatrix} \begin{matrix} R_1 + iR_2 \\ \\ R_3 + (1+i)R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & (1/2)+(3/2)i \\ 0 & 1 & (1/2)-(1/2)i \\ 0 & 0 & 0 \end{bmatrix}$$

So, the eigenvectors for $\lambda = 2$ satisfy $z_1 + ((1/2)+(3/2)i)z_3 = 0$ and $z_2 + ((1/2)-(1/2)i)z_3 = 0$. If we replace the variable z_3 with the parameter α, we see that the eigenvectors of $\lambda = 0$ are $\left\{ \begin{bmatrix} ((-1/2)-(3/2)i)\alpha \\ ((-1/2)+(1/2)i)\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}$, which we can write as Span $\left\{ \begin{bmatrix} (1+3i)/4 \\ (1-i)/4 \\ -1/2 \end{bmatrix} \right\}$. And so have that $\left\{ \begin{bmatrix} (1+3i)/4 \\ (1-i)/4 \\ -1/2 \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace for $\lambda = 2$.

The eigenspace for $\lambda = -2$ is the nullspace of $\begin{bmatrix} 1-\lambda & i & -i \\ -i & -1-\lambda & i \\ i & -i & -\lambda \end{bmatrix} = \begin{bmatrix} 3 & i & -i \\ -i & 1 & i \\ i & -i & 2 \end{bmatrix}$. To find the nullspace, we need to row reduce this matrix:

$$\begin{bmatrix} 3 & i & -i \\ -i & 1 & i \\ i & -i & 2 \end{bmatrix} \begin{matrix} \\ iR_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 3 & i & -i \\ 1 & i & -1 \\ i & -i & 2 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & i & -1 \\ 3 & i & -i \\ i & -i & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - iR_1 \end{matrix} \sim \begin{bmatrix} 1 & i & -1 \\ 0 & -2i & 3-i \\ 0 & 1-i & 2+i \end{bmatrix} \begin{matrix} \\ (1/2)iR_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & i & -1 \\ 0 & 1 & (1/2)+(3/2)i \\ 0 & 1-i & 2+i \end{bmatrix} \begin{matrix} R_1 - iR_2 \\ \\ R_3 + (-1+i)R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & (1/2)-(1/2)i \\ 0 & 1 & (1/2)+(3/2)i \\ 0 & 0 & 0 \end{bmatrix}$$

So, the eigenvectors for $\lambda = -2$ satisfy $z_1 + ((1/2)-(1/2)i)z_3 = 0$ and $z_2 + ((1/2)+(3/2)i)z_3 = 0$. If we replace the variable z_3 with the parameter α, we see that the eigenvectors of $\lambda = -2$ are $\left\{ \begin{bmatrix} ((-1/2)-(1/2)i)\alpha \\ ((-1/2)-(1/2)i)\alpha \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}$, which we can write as Span $\left\{ \begin{bmatrix} (1-i)/4 \\ (1+3i)/4 \\ -1/2 \end{bmatrix} \right\}$. And so have that $\left\{ \begin{bmatrix} (1-i)/4 \\ (1+3i)/4 \\ -1/2 \end{bmatrix} \right\}$ is an orthonormal basis for the eigenspace for $\lambda = -2$.

And so we have that the matrix $U = \begin{bmatrix} (1+i)/\sqrt{8} & (1+3i)/4 & (1-i)/4 \\ (1+i)/\sqrt{8} & (1-i)/4 & (1+3i)/4 \\ 2/\sqrt{8} & -1/2 & -1/2 \end{bmatrix}$

is a unitary matrix such that $U^*FU = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.