Solution to Practice 3w

B2(a) To see if A is unitary, we look at the product A^*A:

$$A^*A = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I$$

So, yes, A is unitary.

B2(b) Since the columns of B have length 2 instead of length 1, B cannot be unitary. (One can quickly see that the columns are orthogonal, though.)

B2(c) To see if C is unitary, we look at the product C^*C:

$$C^*C = \begin{bmatrix} (1-i)/\sqrt{7} & (1-2i)/\sqrt{7} \\ -5/\sqrt{35} & (3-i)/\sqrt{35} \end{bmatrix} \begin{bmatrix} (1+i)/\sqrt{7} & -5/\sqrt{35} \\ (1+2i)/\sqrt{7} & (3+i)/\sqrt{35} \end{bmatrix}$$
$$= \begin{bmatrix} ((1+i)(1-i)+(1-2i)(1+2i))/7 & (-5(1-i)+(1-2i)(3+i))/7\sqrt{5} \\ (-5(1-i)+(3+i)(1-2i))/7\sqrt{5} & (25+(3-i)(3+i))/35 \end{bmatrix} =$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, yes, C is unitary.

B2(d) To see if D is unitary, we look at the product D^*D:

$$D^*D = \begin{bmatrix} (1-i)/\sqrt{6} & -2i/\sqrt{6} \\ (1-i)/\sqrt{3} & -i/\sqrt{3} \end{bmatrix} \begin{bmatrix} (1+i)/\sqrt{6} & (1+i)/\sqrt{3} \\ 2i/\sqrt{6} & i/\sqrt{3} \end{bmatrix}$$
$$= \begin{bmatrix} ((1-i)(1+i)-4i^2)/6 & ((1-i)(1+i)-2i^2)/3\sqrt{2} \\ ((1-i)(1+i)-2i^2)/3\sqrt{2} & ((1-i)(1+i)-i^2)/3 \end{bmatrix} = \begin{bmatrix} 1 & 4/3\sqrt{2} \\ 4/3\sqrt{2} & 1 \end{bmatrix}$$

So we see that D is not unitary, because the columns of D are not orthogonal.

Theorem 9.5.2 (3) Let $C = A + B$, $D = (A+B)^*$, and $E = A^* + B^*$. Then $d_{ij} = \overline{c_{ji}} = \overline{a_{ji} + b_{ji}} = \overline{a_{ji}} + \overline{b_{ji}} = e_{ij}$. This means that $D = E$, so $(A+B)^* = A^* + B^*$.

Theorem 9.5.2 (4) Let $C = \alpha A$, $D = (\alpha A)^*$, and $E = \overline{\alpha}A^*$. Then $d_{ij} = \overline{c_{ji}} = \overline{\alpha a_{ji}} = \overline{\alpha}\ \overline{a_{ji}} = e_{ij}$. This means that $D = E$, so $(\alpha A)^* = \overline{\alpha}A^*$.

Theorem 9.5.2 (5) Let $C = AB$, $D = (AB)^*$, $E = A^*$, and $F = B^*$. Then $d_{ij} = \overline{c_{ji}} = \overline{\sum_{k=1}^n a_{jk}b_{ki}} = \sum_{k=1}^n \overline{a_{jk}b_{ki}} = \sum_{k=1}^n \overline{a_{jk}}\ \overline{b_{ki}} = \sum_{k=1}^n e_{kj}f_{ik} = \sum_{k=1}^n f_{ik}e_{kj} = (FE)_{ij}$. So we have that $D = FE$, which means that $(AB)* =$

B^*A^*.