Lecture 3v

Orthogonality in $\mathbb{C}^n$

(pages 429-430)

Now that we have the inner product correctly defined, we can define orthogonality for complex vector spaces just as we did for real vector spaces.

Definition: Let $\mathbb{V}$ be an inner product space over $\mathbb{C}$, with inner product $\langle\ ,\ \rangle$. Then two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{V}$ are said to be **orthogonal** if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. The set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{V}$ is said to be orthogonal if $\langle \mathbf{v}_j, \mathbf{v}_k \rangle = 0$ for all $j \neq k$. If we also have $\langle \mathbf{v}_j, \mathbf{v}_j \rangle = 1$ for all $1 \leq j \leq n$, then the set is **orthonormal**.

Example: The vectors $\vec{u}_1 = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix}$ are orthogonal, since

$$\begin{aligned} \langle \vec{u}_1, \vec{u}_2 \rangle &= \vec{u}_1 \cdot \overline{\vec{u}_2} = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 1+i \\ 0 \end{bmatrix} \\ &= 1 - i + i + i^2 + 0 = 0 \end{aligned}$$

The vectors $\vec{v}_1 = \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} i \\ 1 \\ i \end{bmatrix}$ are not orthogonal, since

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_2 \rangle &= \vec{v}_1 \cdot \overline{\vec{v}_2} = \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix} \cdot \begin{bmatrix} -i \\ 1 \\ -i \end{bmatrix} \\ &= -i - i^2 + 1 - i + 2i^2 = -2i \neq 0 \end{aligned}$$

As I almost word-for-word copied this definition from Chapter 7, there should be no surprises here. We can also copy the definition for projection, as no changes are needed.

Definition: Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be an orthogonal basis for a subspace $\mathbb{S}$ of $\mathbb{C}^n$. Then the **projection of** $\vec{z} \in \mathbb{C}^n$ **onto** $\mathbb{S}$ is given by

$$\mathrm{proj}_{\mathbb{S}} \vec{z} = \frac{\langle \vec{z}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \cdots + \frac{\langle \vec{z}, \vec{v}_k \rangle}{\langle \vec{v}_k, \vec{v}_k \rangle} \vec{v}_k$$

While this is the same definition we've always used, it is important to remember that our inner product is no longer symmetric, so we must make sure we take our inner product in the correct order. If we use $\langle \vec{v}_1, \vec{z} \rangle$ instead of $\langle \vec{z}, \vec{v}_1 \rangle$, we'll end up with the wrong answer!

Example: Let $\vec{u}_1 = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix}$ be as in the previous example. Then we already know that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbb{S} = \text{Span}\{\vec{u}_1, \vec{u}_2\}$. Let $\vec{z} = \begin{bmatrix} 1+i \\ 1+2i \\ 1+3i \end{bmatrix}$, and let's find $\text{proj}_{\mathbb{S}}\vec{z}$. From the definition, we see that

$$\text{proj}_{\mathbb{S}}\vec{z} = \frac{\langle \vec{z}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle}\vec{u}_1 + \frac{\langle \vec{z}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle}\vec{u}_2$$

Let's compute all the inner products:

$\langle \vec{z}, \vec{u}_1 \rangle = \vec{z} \cdot \overline{\vec{u}_1} = \begin{bmatrix} 1+i \\ 1+2i \\ 1+3i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \\ 2-i \end{bmatrix} = 1+i-i-2i^2+2-i+6i-3i^2 = 8+5i$

$\langle \vec{u}_1, \vec{u}_1 \rangle = 1^2 + 1^2 + 2^2 + 1^2 = 7$

$\langle \vec{z}, \vec{u}_2 \rangle = \vec{z} \cdot \overline{\vec{u}_2} = \begin{bmatrix} 1+i \\ 1+2i \\ 1+3i \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 1+i \\ 0 \end{bmatrix} = 1-i+i-i^2+1+i+2i+2i^2+0 = 1+3i$

$\langle \vec{u}_2, \vec{u}_2 \rangle = 1^2 + 1^2 + 1^2 + (-1)^2 = 4$

So we see that

$$\begin{aligned}
\text{proj}_{\mathbb{S}}\vec{z} &= \tfrac{8+5i}{7} \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} + \tfrac{1+3i}{4} \begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix} \\
&= \tfrac{1}{28} \left(4(8+5i) \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} + 7(1+3i) \begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix} \right) \\
&= \tfrac{1}{28} \begin{bmatrix} 32+20i+7+7i+21i+21i^2 \\ 32i+20i^2+7-7i+21i-21i^2 \\ 64+32i+40i+20i^2+0 \end{bmatrix} \\
&= \tfrac{1}{28} \begin{bmatrix} 18+48i \\ 8+46i \\ 44+72i \end{bmatrix} = \begin{bmatrix} (9/14)+(12/7)i \\ (2/7)+(23/14)i \\ (11/7)+(18/7)i \end{bmatrix}
\end{aligned}$$

And since the definition of $\text{perp}_{\mathbb{S}}\vec{z}$ and the Gram-Schmidt Procedure are all based on the definition of $\text{proj}_{\mathbb{S}}$, these also work the same as they did in $\mathbb{R}^n$, again making sure that we pay attention to the order when we compute any inner products.

Example: Let $\vec{v}_1 = \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} i \\ 1 \\ i \end{bmatrix}$, as in our previous example. Since $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal, the set $\{\vec{v}_1, \vec{v}_2\}$ does not form an orthogonal basis for $\mathbb{S} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. It is a spanning set, though, so we can use it in the Gram-Schmidt Procedure to find an orthogonal basis for $\mathbb{S}$. We start by setting $\vec{w}_1 = \vec{v}_1$ and $\mathbb{S}_1 = \text{Span}\{\vec{w}_1\}$. Then

$$\begin{aligned} \vec{w}_2 &= \text{perp}_{\mathbb{S}_1}\vec{v}_2 \\ &= \vec{v}_2 - \text{proj}_{\mathbb{S}_1}\vec{v}_2 \\ &= \vec{v}_2 - \tfrac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}\vec{w}_1 \end{aligned}$$

Let's compute the inner products we need:

$\langle \vec{v}_2, \vec{w}_1 \rangle = \langle \vec{v}_2, \vec{v}_1 \rangle = \overline{\langle \vec{v}_1, \vec{v}_2 \rangle} = \overline{-2i} = 2i$

$\langle \vec{w}_1, \vec{w}_1 \rangle = 1^2 + 1^2 + 1^2 + (-1)^2 + (-2)^2 = 8$

So we have that

$$\begin{aligned} \vec{w}_2 &= \begin{bmatrix} i \\ 1 \\ i \end{bmatrix} - \left(\tfrac{2i}{8}\right) \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix} \\ &= \begin{bmatrix} i - (1/4)i - (1/4)i^2 \\ 1 - (1/4)i + (1/4)i^2 \\ i + (1/2)i^2 \end{bmatrix} \\ &= \begin{bmatrix} (1/4) + (3/4)i \\ (3/4) - (1/4)i \\ (-1/2) + i \end{bmatrix} \end{aligned}$$

As before, we can pull out the scalar $(1/4)$ from our $\vec{w}_2$ and still have an orthogonal basis vector. And so, by the Gram-Schmidt Procedure, we have that $\mathcal{B} = \left\{ \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix}, \begin{bmatrix} 1+3i \\ 3-i \\ -2+4i \end{bmatrix} \right\}$ is an orthogonal basis for $\mathcal{S}$.