Solution to Practice 3s

B1(a) Let $A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$. First we need to find the eigenvalues for A, and to do that we need to compute the characteristic polynomial $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{bmatrix} \\ &= (1-\lambda)(-3-\lambda) + 5 \\ &= -3 - \lambda + 3\lambda + \lambda^2 + 5 \\ &= 2 + 2\lambda + \lambda^2 \end{aligned}$$

We can use the quadratic formula to find the roots of this equation:

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

This means that the eigenvalues for A are $\lambda = -1 + i$ and $\overline{\lambda} = -1 - i$, and thus that $D = \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix}$, and that $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ is a real canonical form for A

To find a change of coordinates matrix P that brings A into real canonical form, we need to find an eigenvector for one of our eigenvalues, say $\lambda = -1 + i$. To do this, we will go ahead and find the eigenspace for $-1+i$, by finding the nullspace for $\begin{bmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{bmatrix} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}$. And we do this by row reducing our matrix:

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \; R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & -2-i \\ 2-i & -5 \end{bmatrix} \; R_2 + (-2+i)R_1 \sim \begin{bmatrix} 1 & -2-i \\ 0 & 0 \end{bmatrix}.$$

So the eigenvectors for $-1 + i$ satisfy the equation $z_1 + (-2-i)z_2 = 0$, or $z_1 = (2+i)z_2$. If we set $z_2 = 1$, then we see that $\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for $-1+i$. This means that $P \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ is a change of coordinates matrix such that $P^{-1}AP = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$.

A NOTE ON DOUBLE-CHECKING YOUR WORK: An obvious way to double check your answer for this question would be to go ahead and find P^{-1} (its $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$) and do the calculation $P^{-1}AP$ to verify that it does in fact equal $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. This isn't too bad in the 2×2 case, but even in these small matrices we usually end up dealing with fractions, and I certainly think its a pain to find the inverse of a 3×3 matrix. Instead, I trust in the theory, but verify that I really do have the correct eigenvectors. A simple check that $A\vec{z} = \lambda\vec{z}$ is all you need, and you only need to do it for one of the complex eigenvalues (and your real eigenvalue in the 3×3 case). Even though you end up looking at complex vectors, since A only has real entries, $A\vec{z}$ is pretty easy to compute, so $\lambda\vec{z}$ is the only hard part. Obviously, you're preferences (and thoughts as to what constitutes "easy") may be different from mine.

B1(b) Let $A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$. First we need to find the eigenvalues for A, and to do that we need to compute the characteristic polynomial det $(A - \lambda I)$:

$$\begin{aligned} \det\,(A-\lambda I) &= \det \begin{bmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{bmatrix} \\ &= (1-\lambda)(3-\lambda)+5 \\ &= 3-\lambda-3\lambda+\lambda^2+5 \\ &= 8-4\lambda+\lambda^2 \end{aligned}$$

We can use the quadratic formula to find the roots of this equation:

$$\lambda = \frac{4 \pm \sqrt{16-4(1)(8)}}{2(1)} = \frac{4 \pm \sqrt{-16}}{2} = 2 \pm 2i$$

This means that the eigenvalues for A are $\lambda = 2+2i$ and $\overline{\lambda} = 2-2i$, and thus that $D = \begin{bmatrix} 2+2i & 0 \\ 0 & 2-2i \end{bmatrix}$, and that $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is a real canonical form for A

To find a change of coordinates matrix P that brings A into real canonical form, we need to find an eigenvector for one of our eigenvalues, say $\lambda = 2+2i$. To do this, we will go ahead and find the eigenspace for $2+2i$, by finding the nullspace for $\begin{bmatrix} 1-(2+2i) & -5 \\ 1 & 3-(2+2i) \end{bmatrix} = \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix}$. And we do this by row reducing our matrix:

$$\begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & 1-2i \\ -1-2i & -5 \end{bmatrix} R_2 + (1+2i)R_1 \sim \begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}.$$

So the eigenvectors for $2+2i$ satisfy the equation $z_1 + (1-2i)z_2 = 0$, or $z_1 = (-1+2i)z_2$. If we set $z_2 = 1$, then we see that $\begin{bmatrix} -1+2i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is an eigenvector for $2+2i$. This means that $P \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ is a change of coordinates matrix such that $P^{-1}AP = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$.

B1(c) Let $A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$. First we need to find the eigenvalues for A, and to do that we need to compute the characteristic polynomial $\det\,(A - \lambda I)$:

$$\begin{aligned} \det\,(A-\lambda I) &= \det \begin{bmatrix} -\lambda & -2 & 1 \\ 2 & 2-\lambda & -1 \\ 0 & -2 & 2-\lambda \end{bmatrix} \\ &= -\lambda \begin{vmatrix} 2-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} - 2 \begin{vmatrix} -2 & 1 \\ -2 & 2-\lambda \end{vmatrix} + 0 \\ &= -\lambda(4 - 2\lambda - 2\lambda + \lambda^2 - 2) - 2(-4 + 2\lambda + 2) \\ &= -\lambda(2 - 4\lambda + \lambda^2) - 2(-2 + 2\lambda) \\ &= -2\lambda + 4\lambda^2 - \lambda^3 + 4 - 4\lambda \\ &= 4 - 6\lambda + 4\lambda^2 - \lambda^3 \end{aligned}$$

A little "guess and check" shows that $\lambda = 2$ is a root of this polynomial. If we factor out $(2 - \lambda)$, we are left with $2 - 2\lambda + \lambda^2$, and we can use the quadratic formula to find the roots of this equation:

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

This means that the eigenvalues for A are $\mu = 2$, $\lambda = 1 + i$ and $\overline{\lambda} = 1 - i$, and thus that $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$, and that $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ is a real canonical form for A

To find a change of coordinates matrix P that brings A into real canonical form, we need to find an eigenvector for $\mu = 2$ and for one of our eigenvalues, say $\lambda = 1 + i$. To do this, we will go ahead and find the eigenspaces.

For 2, the eigenspace is the nullspace for $\begin{bmatrix} -2 & -2 & 1 \\ 2 & 2-2 & -1 \\ 0 & -2 & 2-2 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 1 \\ 2 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.
And we find this by row reducing our matrix:

$$\begin{bmatrix} -2 & -2 & 1 \\ 2 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0-1 & \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors for $\mu = 2$ satisfy the equations $2z_1 - z_3 = 0$ and $z_2 = 0$. If we set $z_1 = 1$, then we see that $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is an eigenvector for $\mu = 2$.

For $1+i$, the eigenspace is the nullspace for $\begin{bmatrix} -1-i & -2 & 1 \\ 2 & 2-1-i & -1 \\ 0 & -2 & 2-1-i \end{bmatrix} =$ $\begin{bmatrix} -1-i & -2 & 1 \\ 2 & 1-i & -1 \\ 0 & -2 & 1-i \end{bmatrix}$. And we find this by row reducing our matrix:

$$\begin{bmatrix} -1-i & -2 & 1 \\ 2 & 1-i & -1 \\ 0 & -2 & 1-i \end{bmatrix} \begin{matrix} (1/2)(-1+i)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1-i & -\frac{1}{2}+\frac{1}{2}i \\ 2 & 1-i & -1 \\ 0 & -2 & 1-i \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1-i & -\frac{1}{2}+\frac{1}{2}i \\ 0 & -1+i & -i \\ 0 & -2 & 1-i \end{bmatrix} (1/2)(-1-i)R_2 \sim \begin{bmatrix} 1 & 1-i & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & -2 & 1-i \end{bmatrix} \begin{matrix} R_1 + (-1+i)R_2 \\ \\ R_3 + 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2}-\frac{1}{2}i \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors for $1+i$ satisfy the equations $z_1 + (-1/2)(1+i)z_3 = 0$ and $z_2 + (-1/2)(1-i)z_3 = 0$. If we set $z_3 = 2$, then we see that $\begin{bmatrix} 1+i \\ 1-i \\ 2 \end{bmatrix} =$ $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector for $1+i$.

And we now see that $P \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ is a change of coordinates matrix such that $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

B1(d) Let $A = \begin{bmatrix} -1 & 2 & -2 \\ -2 & -1 & -1 \\ 4 & -2 & 5 \end{bmatrix}$. First we need to find the eigenvalues for A, and to do that we need to compute the characteristic polynomial $\det(A - \lambda I)$:

$$\begin{aligned}
\det\,(A-\lambda I) &= \det \begin{bmatrix} -1-\lambda & 2 & -2 \\ -2 & -1-\lambda & -1 \\ 4 & -2 & 5-\lambda \end{bmatrix} \\
&= (-1-\lambda)\begin{vmatrix} -1-\lambda & -1 \\ -2 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & -1 \\ 4 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & -1-\lambda \\ 4 & -2 \end{vmatrix} \\
&(-1-\lambda)(-5+\lambda-5\lambda+\lambda^2-2) - 2(-10+2\lambda+4) - 2(4+4+4\lambda) \\
&= (-1-\lambda)(-7-4\lambda+\lambda^2) - 2(-6+2\lambda) - 2(8+4\lambda) \\
&= 7+4\lambda-\lambda^2+7\lambda+4\lambda^2-\lambda^3+12-4\lambda-16-8\lambda \\
&= 3-\lambda+3\lambda^2-\lambda^3
\end{aligned}$$

A little "guess and check" shows that $\lambda = 3$ is a root of this polynomial. If we factor out $(3-\lambda)$, we are left with $1+\lambda^2$, which factors as $(i-\lambda)(-i-\lambda)$. This means that the eigenvalues for A are $\mu = 3$, $\lambda = i$ and $\overline{\lambda} = -i$, and thus that $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$, and that $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ is a real canonical form for A

To find a change of coordinates matrix P that brings A into real canonical form, we need to find an eigenvector for $\mu = 3$ and for one of our eigenvalues, say $\lambda = i$. To do this, we will go ahead and find the eigenspaces.

For 3, the eigenspace is the nullspace for $\begin{bmatrix} -1-3 & 2 & -2 \\ -2 & -1-3 & -1 \\ 4 & -2 & 5-3 \end{bmatrix} = \begin{bmatrix} -4 & 2 & -2 \\ -2 & -4 & -1 \\ 4 & -2 & 2 \end{bmatrix}$.
And we find this by row reducing our matrix:

$$\begin{bmatrix} -4 & 2 & -2 \\ -2 & -4 & -1 \\ 4 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors for $\mu = 3$ satisfy the equations $z_1 + (1/2)z_3 = 0$ and $z_2 = 0$.
If we set $z_3 = -2$, then we see that $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ is an eigenvector for $\mu = 2$.

For i, the eigenspace is the nullspace for $\begin{bmatrix} -1-i & 2 & -2 \\ -2 & -1-i & -1 \\ 4 & -2 & 5-i \end{bmatrix}$. And we find this by row reducing our matrix:

$$\begin{bmatrix} -1-i & 2 & -2 \\ -2 & -1-i & -1 \\ 4 & -2 & 5-i \end{bmatrix} \begin{matrix} (1/2)(-1+i)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1+i & 1-i \\ -2 & -1-i & -1 \\ 4 & -2 & 5-i \end{bmatrix} \begin{matrix} \\ R_2+2R_1 \\ R_3-4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1+i & 1-i \\ 0 & -3+i & 1-2i \\ 0 & 2-4i & 1+3i \end{bmatrix} \; (1/10)(-3-i)R_2 \;\sim \begin{bmatrix} 1 & -1+i & 1-i \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 2-4i & 1+3i \end{bmatrix} \begin{matrix} R_1+(1-i)R_2 \\ \\ R_3+(-2+4i)R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors for i satisfy the equations $z_1 + z_3 = 0$ and $z_2 + (-1/2)(1-i)z_3 = 0$. If we set $z_3 = 2$, then we see that $\begin{bmatrix} -2 \\ 1-i \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector for i.

And we now see that $P \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$ is a change of coordinates matrix such that $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

B1(e) Let $A = \begin{bmatrix} 6 & 0 & -4 \\ 0 & 1 & 1 \\ 8 & -1 & -5 \end{bmatrix}$. First we need to find the eigenvalues for A, and to do that we need to compute the characteristic polynomial det $(A - \lambda I)$:

$$\begin{aligned} \det(A-\lambda I) &= \det \begin{bmatrix} 6-\lambda & 0 & -4 \\ 0 & 1-\lambda & 1 \\ 8 & -1 & -5-\lambda \end{bmatrix} \\ &= (6-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -1 & -5-\lambda \end{vmatrix} + 0 + 8 \begin{vmatrix} 0 & -4 \\ 1-\lambda & 1 \end{vmatrix} \\ &= (6-\lambda)(-5-\lambda+5\lambda+\lambda^2+1) + 8(4-4\lambda) \\ &= (6-\lambda)(-4+4\lambda+\lambda^2) + 8(4-4\lambda) \\ &= -24+24\lambda+6\lambda^2+4\lambda-4\lambda^2-\lambda^3+32-32\lambda \\ &= 8-4\lambda+2\lambda^2-\lambda^3 \end{aligned}$$

A little "guess and check" shows that $\lambda = 2$ is a root of this polynomial. If we factor out $(2-\lambda)$, we are left with $4+\lambda^2$, which factors as $(2i-\lambda)(-2i-\lambda)$. This means that the eigenvalues for A are $\mu = 2$, $\lambda = 2i$ and $\overline{\lambda} = -2i$, and thus that $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2i & \\ 0 & 0 & -2i \end{bmatrix}$, and that $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$ is a real canonical form for A

To find a change of coordinates matrix P that brings A into real canonical form, we need to find an eigenvector for $\mu = 2$ and for one of our eigenvalues, say $\lambda = 2i$. To do this, we will go ahead and find the eigenspaces.

For 2, the eigenspace is the nullspace for $\begin{bmatrix} 6-2 & 0 & -4 \\ 0 & 1-2 & 1 \\ 8 & -1 & -5-2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -4 \\ 0 & -1 & 1 \\ 8 & -1 & -7 \end{bmatrix}$. And we find this by row reducing our matrix:

$$\begin{bmatrix} 4 & 0 & -4 \\ 0 & -1 & 1 \\ 8 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors for $\mu = 2$ satisfy the equations $z_1 - z_3 = 0$ and $z_2 - z_3 = 0$. If we set $z_1 = 1$, then we see that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\mu = 2$.

For $2i$, the eigenspace is the nullspace for $\begin{bmatrix} 6-2i & 0 & -4 \\ 0 & 1-2i & 1 \\ 8 & -1 & -5-2i \end{bmatrix}$. And we find this by row reducing our matrix:

$$\begin{bmatrix} 6-2i & 0 & -4 \\ 0 & 1-2i & 1 \\ 8 & -1 & -5-2i \end{bmatrix} \begin{matrix} (1/40)(6+2i)R_1 \\ (1/5)(1+2i)R_2 \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -\frac{3}{5}-\frac{1}{5}i \\ 0 & 1 & \frac{1}{5}+\frac{2}{5}i \\ 8 & -1 & -5-2i \end{bmatrix} \ R_3 + 8R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{3}{5}-\frac{1}{5}i \\ 0 & 1 & \frac{1}{5}+\frac{2}{5}i \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors for $2i$ satisfy the equations $z_1 + (-1/5)(3+i)z_3 = 0$ and $z_2 + (-1/5)(-1-2i)z_3 = 0$. If we set $z_3 = 5$, then we see that $\begin{bmatrix} 3+i \\ -1-2i \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} + i \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ is an eigenvector for $1+i$.

And we now see that $P \begin{bmatrix} 1 & 3 & 1 \\ 1 & -1 & -2 \\ 1 & 5 & 0 \end{bmatrix}$ is a change of coordinates matrix such that $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$.