Solution to Practice 3r

D4 Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $\mathbb{R}^3$. We want to show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is also a basis for $\mathbb{C}^3$, so we need to show that it is linearly independent and that it is a spanning set for $\mathbb{C}^3$.

linear independence: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ be such that

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}$$

and let $\alpha_1 = a_1 + b_1 i$, $\alpha_2 = a_2 + b_2 i$, and $\alpha_3 = a_3 + b_3 i$, for $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. Then we can rewrite our equation as

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + i(b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3) = \vec{0} + i\vec{0}$$

This means that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$$

and

$$b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 = \vec{0}$$

in the real numbers. Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent in $\mathbb{R}$, we must have that $a_1 = a_2 = a_3 = 0$ and $b_1 = b_2 = b_3 = 0$. And this means that $\alpha_1 = 0 + 0i = 0$, $\alpha_2 = 0 + 0i = 0$, and $\alpha_3 = 0 + 0i = 0$. And so we see that the only solution to the equation

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0}$$

is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which means that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

span: Let $\vec{z} \in \mathbb{C}^3$, and let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be such that $\vec{z} = \vec{x} + i\vec{y}$. Since $\vec{x}, \vec{y} \in \mathbb{R}^3$, there must be scalars $s_1, s_2, s_3, t_1, t_2, t_3 \in \mathbb{R}$ such that

$$s_1\vec{v}_1 + s_2\vec{v}_2 + s_3\vec{v}_3 = \vec{x} \quad \text{and} \quad t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{y}$$

Then we have that

$$\begin{aligned}(s_1 + it_1)\vec{v}_1 + (s_2 + it_2)\vec{v}_2 + (s_3 + it_3)\vec{v}_3 &= s_1\vec{v}_1 + s_2\vec{v}_2 + s_3\vec{v}_3 + i(t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3) \\ &= \vec{x} + i\vec{y} \\ &= \vec{z}\end{aligned}$$

And so we see that for any $\vec{z} \in \mathbb{C}^3$, $\vec{z} \in \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, which means that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a spanning set for $\mathbb{C}^3$.