Solution to Practice 3q

A1(a) Let $A = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$. To find the eigenvalues of A, we need to first find the characteristic polynomial.

$$\begin{aligned} \det\,(A - \lambda I) &= \det \begin{bmatrix} -\lambda & 4 \\ -1 & -\lambda \end{bmatrix} \\ &= (-\lambda)^2 + 4 \\ &= \lambda^2 + 4 \\ &= (\lambda - 2i)(\lambda + 2i) \end{aligned}$$

So, our eigenvalues are $\lambda = 2i$ and $\overline{\lambda} = -2i$. The eigenvectors for $\lambda = 2i$ are the nullspace of $\begin{bmatrix} -2i & 4 \\ -1 & -2i \end{bmatrix}$, which we find by row reducing this matrix:

$$\begin{bmatrix} -2i & 4 \\ -1 & -2i \end{bmatrix} \; R_1 \updownarrow R_2 \sim \begin{bmatrix} -1 & -2i \\ -2i & 4 \end{bmatrix} \; -R_1$$
$$\sim \begin{bmatrix} 1 & 2i \\ -2i & 4 \end{bmatrix} \; R_2 + 2iR_1 \sim \begin{bmatrix} 1 & 2i \\ 0 & 0 \end{bmatrix}$$

So, the eigenvectors satisfy the equation $z_1 + 2iz_2 = 0$, which is the same as the set $\operatorname{Span}\left\{ \begin{bmatrix} -2i \\ 1 \end{bmatrix} \right\}$.

Using Theorem 9.4.1, we know that the eigenvectors for $\overline{\lambda} = -2i$ are $\operatorname{Span}\left\{ \begin{bmatrix} \overline{-2i} \\ \overline{1} \end{bmatrix} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 2i \\ 1 \end{bmatrix} \right\}$.

And so we have that $P = \begin{bmatrix} -2i & 2i \\ 1 & 1 \end{bmatrix}$ is an invertible matrix and $D = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$ is a diagonal matrix such that $P^{-1}AP = D$.

A1(b) Let $A = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$. To find the eigenvalues of A, we need to first find the characteristic polynomial.

$$\begin{aligned} \det\,(A - \lambda I) &= \det \begin{bmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{bmatrix} \\ &= (-1-\lambda)(-3-\lambda) + 2 \\ &= 3 + \lambda + 3\lambda + \lambda^2 + 2 \\ &= \lambda^2 + 4\lambda + 5 \end{aligned}$$

We can use the quadratic formula to find the roots of this equation:

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

So, our eigenvalues are $\lambda = -2+i$ and $\overline{\lambda} = -2-i$. The eigenvectors for $\lambda = -2+i$ are the nullspace of $\begin{bmatrix} -1-(-2+i) & 2 \\ -1 & -3-(-2+i) \end{bmatrix} = \begin{bmatrix} 1-i & 2 \\ -1 & -1+i \end{bmatrix}$, which we find by row reducing this matrix:

$$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix} \begin{matrix} (1/2)(1+i)R_1 \\ \end{matrix} \sim \begin{bmatrix} 1 & 1+i \\ -1 & -1-i \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$$

So, the eigenvectors satisfy the equation $z_1 + (1+i)z_2 = 0$, which is the same as the set Span $\left\{ \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \right\}$.

Using Theorem 9.4.1, we know that the eigenvectors for $\overline{\lambda} = -2-i$ are Span $\left\{ \begin{bmatrix} \overline{-1-i} \\ \overline{1} \end{bmatrix} \right\} =$ Span $\left\{ \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\}$.

And so we have that $P = \begin{bmatrix} -1-i & -1+i \\ 1 & 1 \end{bmatrix}$ is an invertible matrix and $D = \begin{bmatrix} -2+i & 0 \\ 0 & -2-i \end{bmatrix}$ is a diagonal matrix such that $P^{-1}AP = D$.

A1(c) Let $A = \begin{bmatrix} 2 & 2 & -1 \\ -4 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$. To find the eigenvalues of A, we need to first find the characteristic polynomial.

$$\begin{aligned}
\det\ (A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 2 & -1 \\ -4 & 1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{bmatrix} \\
&= (2-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} -4 & 2 \\ 2 & -1-\lambda \end{vmatrix} - \begin{vmatrix} -4 & 1-\lambda \\ 2 & 2 \end{vmatrix} \\
&= (2-\lambda)((1-\lambda)(-1-\lambda) - 4) - 2((-4)(-1-\lambda) - 4) - (-8 - 2(1-\lambda)) \\
&= (2-\lambda)(-1-\lambda+\lambda+\lambda^2-4) - 2(4+4\lambda-4) - (-8-2+2\lambda) \\
&= (2-\lambda)(\lambda^2+5) - 2(4\lambda) - (-10+2\lambda) \\
&= 2\lambda^2 - 10 - \lambda^3 + 5\lambda - 8\lambda + 10 + 2\lambda \\
&= -5\lambda + 2\lambda^2 - \lambda^3 \\
&= -\lambda(5 - 2\lambda + \lambda^2)
\end{aligned}$$

Because $-\lambda$ is a factor of the characteristic polynomial, we know that $\lambda = 0$ is an eigenvalue of A. To find the other eigenvalues of A, we use the quadratic formula to find the roots of $5 - 2\lambda + \lambda^2$:

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

So, the eigenvalues of A are $\lambda = 0, 1+2i, 1-2i$. Now let's find their eigenspaces.

The eigenspace for $\lambda = 0$ is the nullspace of $A - \lambda I = A - 0I = A$, and to find the nullspace of A we need to row reduce A. $\begin{bmatrix} 2 & 2 & -1 \\ -4 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim$
$\begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So the eigenvectors of $\lambda = 0$ are the the vectors that satisfy $2z_1 - z_3 = 0$ and $z_2 = 0$, which we can write as Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

The eigenspace for $\lambda = 1+2i$ is the nullspace of $A - \lambda I = \begin{bmatrix} 2-(1+2i) & 2 & -1 \\ -4 & 1-(1+2i) & 2 \\ 2 & 2 & -1-(1+2i) \end{bmatrix} =$
$\begin{bmatrix} 1-2i & 2 & -1 \\ -4 & -2i & 2 \\ 2 & 2 & -2-2i \end{bmatrix}$, which we find by row reducing this matrix.

$$\begin{bmatrix} 1-2i & 2 & -1 \\ -4 & -2i & 2 \\ 2 & 2 & -2-2i \end{bmatrix} \quad R_1 \updownarrow R_3 \quad \sim \begin{bmatrix} 2 & 2 & -2-2i \\ -4 & -2i & 2 \\ 1-2i & 2 & -1 \end{bmatrix} \quad (1/2)R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -1-1i \\ -4 & -2i & 2 \\ 1-2i & 2 & -1 \end{bmatrix} \quad \begin{matrix} \\ R_2 + 4R_1 \\ R_3 + (-1+2i)R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & 1 & -1-i \\ 0 & 4-2i & -2-4i \\ 0 & 1+2i & 2-i \end{bmatrix} \quad (1/20)(4+2i)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1-i \\ 0 & 1 & -i \\ 0 & 1+2i & 2-i \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ \\ R_3 + (-1-2i)R_2 \end{matrix} \quad \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors of $\lambda = 1+2i$ are the the vectors that satisfy $z_1 - z_3 = 0$ and $z_2 - iz_3 = 0$. If we replace the variable z_3 with the parameter α, we see that the general solution to this system is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ i\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} =$$

which we can write as Span $\left\{ \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} \right\}$.

By Theorem 9.4.1, the eigenspace for $\lambda = 1-2i$ is Span $\left\{ \begin{bmatrix} \overline{1} \\ \overline{i} \\ \overline{1} \end{bmatrix} \right\}$ = Span $\left\{ \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} \right\}$.

So we have that $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 0$, $\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1 + 2i$, and $\begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1 - 2i$. And this means that the matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & i & -i \\ 2 & 1 & 1 \end{bmatrix}$ is such that $P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}$.

A1(d) Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix}$. To find the eigenvalues of A, we need to first find the characteristic polynomial.

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 1 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{bmatrix} \\
&= -2 \begin{vmatrix} 1 & -1 \\ -1 & 2-\lambda \end{vmatrix} + (1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 3 & 2-\lambda \end{vmatrix} - 0 \\
&= -2(2-\lambda-1) + (1-\lambda)(4-2\lambda-2\lambda+\lambda^2+3) \\
&= -2(1-\lambda) + (1-\lambda)(7-4\lambda+\lambda^2) \\
&= (1-\lambda)(-2+7-4\lambda+\lambda^2) \\
&= (1-\lambda)(5-4\lambda+\lambda^2)
\end{aligned}$$

Because $1 - \lambda$ is a factor of the characteristic polynomial, we know that $\lambda = 1$ is an eigenvalue of A. To find the other eigenvalues of A, we use the quadratic formula to find the roots of $5 - 4\lambda + \lambda^2$:

$$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

So, the eigenvalues of A are $\lambda = 2, 2+i, 2-i$. Now let's find their eigenspaces.

The eigenspace for $\lambda = 1$ is the nullspace of $A - \lambda I = \begin{bmatrix} 2-1 & 1 & -1 \\ 2 & 1-1 & 0 \\ 3 & -1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}$, which we find by row reducing this matrix.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors of $\lambda = 1$ are the the vectors that satisfy $z_1 = 0$ and $z_2 - z_3 = 0$, which we can write as Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

The eigenspace for $\lambda = 2+i$ is the nullspace of $A - \lambda I = \begin{bmatrix} 2-(2+i) & 1 & -1 \\ 2 & 1-(2+i) & 0 \\ 3 & -1 & 2-(2+i) \end{bmatrix} = \begin{bmatrix} -i & 1 & -1 \\ 2 & -1-i & 0 \\ 3 & -1 & -i \end{bmatrix}$, which we find by row reducing this matrix.

$$\begin{bmatrix} -i & 1 & -1 \\ 2 & -1-i & 0 \\ 3 & -1 & -i \end{bmatrix} \begin{matrix} iR_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & i & -i \\ 2 & -1-i & 0 \\ 3 & -1 & -i \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & i & -i \\ 0 & -1-3i & 2i \\ 0 & -1-3i & 2i \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & i & -i \\ 0 & -1-3i & 2i \\ 0 & 0 & 0 \end{bmatrix} (1/10)(-1+3i)R_2$$
$$\sim \begin{bmatrix} 1 & i & -i \\ 0 & 1 & -\frac{3}{5} - \frac{1}{5}i \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - iR_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{5}(-1-2i) \\ 0 & 1 & \frac{1}{5}(-3-i) \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors of $\lambda = 2+i$ are the the vectors that satisfy $z_1 + (1/5)(-1-2i)z_3 = 0$ and $z_2 + (1/5)(-3-i)z_3 = 0$. If we replace the variable z_3 with the parameter α, we see that the general solution to this system is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} (1/5)(1+2i)\alpha \\ (1/5)(3+i)\alpha \\ \alpha \end{bmatrix} = (1/5)\alpha \begin{bmatrix} 1+2i \\ 3+i \\ 5 \end{bmatrix} =$$

which we can write as Span $\left\{ \begin{bmatrix} 1+2i \\ 3+i \\ 5 \end{bmatrix} \right\}$.

By Theorem 9.4.1, the eigenspace for $\lambda = 2 - i$ is $\text{Span}\left\{\begin{bmatrix} \overline{1+2i} \\ \overline{3+i} \\ \overline{5} \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1-2i \\ 3-i \\ 5 \end{bmatrix}\right\}$.

So we have that $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1$, $\begin{bmatrix} 1+2i \\ 3+i \\ 5 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 2+i$, and $\begin{bmatrix} 1-2i \\ 3-i \\ 5 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 2-i$. This means that the matrix $P = \begin{bmatrix} 0 & 1+2i & 1-2i \\ 1 & 3+i & 3-i \\ 1 & 5 & 5 \end{bmatrix}$ is such that $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}$.