Lecture 3o

Complex Multiplication as a Matrix Mapping

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When we introduced the concept of graphing complex numbers on the complex plane, we did so from the point of view of considering complex numbers as a two-dimensional vector space over $\mathbb{R}$. The main point of this was to introduce polar notation for complex numbers, but we first noted that addition of complex numbers followed the usual parallelogram rule for vector addition, and I made mention of the fact that there was a geometrical interpretation of complex multiplication as well. It is a this point that I want to go back and figure out just what the geometrical interpretation is. The inspiration comes from the polar definition of complex number multiplication:

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Let us consider for a moment the case when r_1 and r_2 are both 1. Then we have

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

In this case, to multiply our complex numbers we simply need to add their arguments. Another way to think of this is that we would start at $e^{i\theta_1}$ and rotate an additional θ_2 around the origin to get to $e^{i(\theta_1+\theta_2)}$. That is to say, that multiplying by $e^{i\theta_2}$ is the same as rotating by θ_2.

If we let r_1 and r_2 be numbers other than 1, the situation becomes only a bit more complicated. First of all, we still see that we get $r_1 e^{i(\theta_1+\theta_2)}$ from $r_1 e^{i\theta_1}$ by rotating by θ_2. And then, to get from $r_1 e^{i(\theta_1+\theta_2)}$ to $r_1 r_2 e^{i(\theta_1+\theta_2)}$, we simply need to multiply by the scalar r_2, which is the same thing as dilating (or contracting if $r_2 < 1$) by r_2.

So we see that, geometrically, we can think of multiplication by a complex number as a combination of a rotation and a contraction. As these actions are both linear mapping over $\mathbb{R}^2$, their composition is also a linear mapping, and as such there must be a matrix in $M(2,2)$ that represents it.

To find this matrix, let's fix a complex number $\alpha = a + ib = \begin{bmatrix} a \\ b \end{bmatrix}$, and define the linear mapping $L_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by $L_\alpha(x,y) = (ax - by, bx + ay)$. Since $(a+bi)(x+yi) = (ax-by) + (bx+ay)i$, this is the linear mapping for multiplying $\begin{bmatrix} x \\ y \end{bmatrix}$ by $\begin{bmatrix} a \\ b \end{bmatrix}$. To find its standard matrix, we first need to see that $L_\alpha(1,0) = (a,b)$ and $L_\alpha(0,1) = (-b,a)$. From this we get that

$$[L_\alpha] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The textbook refers to this matrix as M_α.

Example: Let $\alpha = 3 - 4i$. Find M_α, and use it to calculate $(3-4i)(2+5i)$.

We have that $M_\alpha = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$, and

$$(3-4i)(2+5i) = M_\alpha \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 7 \end{bmatrix} = 26 + 7i$$