Lecture 3l

Subspaces

Definition: Suppose that $\mathbb{V}$ is a vector space over $\mathbb{C}$, and that $\mathbb{U}$ is a subset of $\mathbb{V}$. If $\mathbb{U}$ is a vector space over $\mathbb{C}$ using the same definition of addition and scalar multiplication as $\mathbb{V}$, then $\mathbb{U}$ is called a **subspace** of $\mathbb{V}$.

As we found in $\mathbb{R}$, any subset (but not necessarily a vector space) $\mathbb{W}$ of a vector space $\mathbb{V}$ will automatically satisfy properties V2, V5, V7, V8, V9, and V10. So, if we want to prove that $\mathbb{W}$ is itself a vector space, we only need to look at properties V1, V4, V5, and V6. We can easily use the same proof as in $\mathbb{R}$, to show that $0\mathbf{v} = \mathbf{0}$ and $(-1)\mathbf{v} = -\mathbf{v}$ in $\mathbb{C}$ as well. This means that properties V4 and V5 will follow from property V6 (closure under scalar multiplication). And so, as before, we get the following alternate definition of a subspace.

Definition: Suppose that $\mathbb{V}$ is a vector space over $\mathbb{C}$. Then $\mathbb{U}$ is a subspace of $\mathbb{V}$ if is satisfies the following three properties:

S0: $\mathbb{U}$ is a non-empty subset of $\mathbb{V}$
S1: $\mathbf{w} + \mathbf{z} \in \mathbb{U}$ for all $\mathbf{w}, \mathbf{z} \in \mathbb{U}$ ($\mathbb{U}$ is closed under addition)
S2: $\alpha\mathbf{z} \in \mathbb{U}$ for all $\mathbf{z} \in \mathbb{U}$ and $\alpha \in \mathbb{C}$ ($\mathbb{U}$ is closed under scalar multiplication)

Example: Show that the set $\mathbb{U} = \left\{ \begin{bmatrix} z \\ 2z \end{bmatrix} \mid z \in \mathbb{C} \right\}$ is a subspace of $\mathbb{C}^2$.

We need to verify the three defining properties.

S0: We note that $\begin{bmatrix} z \\ 2z \end{bmatrix} \in \mathbb{C}^2$ for all $z \in \mathbb{C}$, so $\mathbb{U}$ is a subset of $\mathbb{C}^2$. To see that it is non-empty, we note that $2(0) = 0$, so $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{U}$.

S1: Let $\vec{w}, \vec{z} \in \mathbb{U}$. Then we have that $\vec{w} = \begin{bmatrix} w \\ 2w \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} z \\ 2z \end{bmatrix}$ for some $w, z \in \mathbb{C}$. Then we have that

$$\begin{aligned} \vec{w} + \vec{z} &= \begin{bmatrix} w \\ 2w \end{bmatrix} + \begin{bmatrix} z \\ 2z \end{bmatrix} \\ &= \begin{bmatrix} w + z \\ 2w + 2z \end{bmatrix} \\ &= \begin{bmatrix} w + z \\ 2(w + z) \end{bmatrix} \end{aligned}$$

and since $w + z \in \mathbb{C}$, we see that $\vec{w} + \vec{z} \in \mathbb{U}$.

S2: Let $\vec{z} \in \mathbb{U}$ (so $\vec{z} = \begin{bmatrix} z \\ 2z \end{bmatrix}$), and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} \alpha\vec{z} &= \alpha \begin{bmatrix} z \\ 2z \end{bmatrix} \\ &= \begin{bmatrix} \alpha z \\ \alpha(2z) \end{bmatrix} \\ &= \begin{bmatrix} \alpha z \\ 2(\alpha z) \end{bmatrix} \end{aligned}$$

and since $\alpha z \in \mathbb{C}$, we see that $\alpha\vec{z} \in \mathbb{U}$.

Example: Let $C(2,2)$ be the set of all 2×2 matrices with entries from the complex numbers, and let $\mathcal{A} = \left\{ \begin{bmatrix} z_1 & z_2 \\ z_1 & z_2 \end{bmatrix} \mid z_1, z_2 \in \mathbb{C} \right\}$. Then $\mathcal{A}$ is a subspace of $C(2,2)$. To prove this, we check the three properties:

S0: $\begin{bmatrix} z_1 & z_2 \\ z_1 & z_2 \end{bmatrix} \in C(2,2)$ for all $z_1, z_2 \in \mathbb{C}$, so $\mathcal{A}$ is a subset of $C(2,2)$. To see that $\mathcal{A}$ is non-empty, we can set $z_1 = z_2 = 0$, and see that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{A}$.

S1: Let $A, B \in \mathcal{A}$, say $A = \begin{bmatrix} a_1 & a_2 \\ a_1 & a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_1 & b_2 \end{bmatrix}$. Then we have

$$A + B = \begin{bmatrix} a_1 & a_2 \\ a_1 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_1 + b_1 & a_2 + b_2 \end{bmatrix}$$

and since $a_1 + b_1 \in \mathbb{C}$ and $a_2 + b_2 \in \mathbb{C}$, we see that $A + B \in \mathcal{A}$.

S2: Let $A \in \mathcal{A}$ (say $A = \begin{bmatrix} a_1 & a_2 \\ a_1 & a_2 \end{bmatrix}$) and $\alpha \in \mathbb{C}$. Then

$$\alpha A = \alpha \begin{bmatrix} a_1 & a_2 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_1 & \alpha a_2 \end{bmatrix}$$

and since $\alpha a_1 \in \mathbb{C}$ and $\alpha a_2 \in \mathbb{C}$, we see that $\alpha A \in \mathcal{A}$.

Example: To see that the set $\mathbb{W} = \left\{ \begin{bmatrix} z \\ z^2 \end{bmatrix} \mid z \in \mathbb{C} \right\}$ is not a subspace of $\mathbb{C}^2$, consider that $\begin{bmatrix} i \\ -1 \end{bmatrix} \in \mathbb{W}$ and $\begin{bmatrix} -i \\ -1 \end{bmatrix} \in \mathbb{W}$, but $\begin{bmatrix} i \\ -1 \end{bmatrix} + \begin{bmatrix} -i \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -2 \end{bmatrix} \notin \mathbb{W}$. As such, property S1 fails, so $\mathbb{W}$ is not a subspace.