Lecture 3j

Vector Spaces over $\mathbb{C}$

(pages 411-412)

Now that we know how to add and multiply complex numbers, we can look at vectors from $\mathbb{C}^n$. Just as in $\mathbb{R}^n$, a vector in $\mathbb{C}^n$ will be an ordered list of n complex numbers. And we will define addition and scalar multiplication on vectors componentwise.

Definition: The vector space $\mathbb{C}^n$ is defined to be the set

$$\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \mid z_1, \ldots, z_n \in \mathbb{C} \right\}$$

with addition of vectors defined by

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} z_1 + w_1 \\ \vdots \\ z_n + w_n \end{bmatrix}$$

and scalar multiplication of vectors defined by

$$s \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} sz_1 \\ \vdots \\ sz_n \end{bmatrix}$$

for all $s \in \mathbb{C}$. (Notice that our scalars are now from $\mathbb{C}$, not $\mathbb{R}$.)

We also extend the notion of a complex conjugate to vectors in $\mathbb{C}^n$ as well.

Definition: The **complex conjugate** of $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$ is defined to be $\overline{\vec{z}} = \begin{bmatrix} \overline{z_1} \\ \vdots \\ \overline{z_n} \end{bmatrix}$.

Example:

(1) $\begin{bmatrix} 2+3i \\ i \\ -1-5i \end{bmatrix} + \begin{bmatrix} 7-5i \\ 6 \\ 4+i \end{bmatrix} = \begin{bmatrix} (2+3i)+(7-5i) \\ (i)+(6) \\ (-1-5i)+(4+i) \end{bmatrix} = \begin{bmatrix} 9-2i \\ 6+i \\ 3-4i \end{bmatrix}$

(2) $(3-i)\begin{bmatrix} 1+i \\ 1-2i \\ 5-i \end{bmatrix} = \begin{bmatrix} (3-i)(1+i) \\ (3-i)(1-2i) \\ (3-i)(5-i) \end{bmatrix} = \begin{bmatrix} 3+3i-i-i^2 \\ 3-3i-3i+2i^2 \\ 15-3i-5i+i^2 \end{bmatrix} = \begin{bmatrix} 4+2i \\ 1-4i \\ 14-8i \end{bmatrix}$

(3) $(2+4i)\begin{bmatrix} -3-3i \\ 5+4i \end{bmatrix} - 3i\begin{bmatrix} 4 \\ 7+i \end{bmatrix} = \begin{bmatrix} -6-6i-12i-12i^2 \\ 10+8i+20i+16i^2 \end{bmatrix} + \begin{bmatrix} -12i \\ -21i-3i^2 \end{bmatrix} =$
$\begin{bmatrix} 6-30i \\ -27+25i \end{bmatrix}$

(4) If $\vec{z} = \begin{bmatrix} 2+3i \\ 1-5i \\ 2 \\ -3i \end{bmatrix}$, then $\overline{\vec{z}} = \begin{bmatrix} \overline{2+3i} \\ \overline{1-5i} \\ \overline{2} \\ \overline{-3i} \end{bmatrix} = \begin{bmatrix} 2-3i \\ 1+5i \\ 2 \\ 3i \end{bmatrix}$

As always, our definitions of addition and scalar multiplication will satisfy the vector space properties, with $\mathbb{R}$ replaced by $\mathbb{C}$.

Definition: A **vector space over** $\mathbb{C}$ is a set $\mathbb{V}$ together with an operation of **addition**, usually denoted $\mathbf{x}+\mathbf{y}$ for any $\mathbf{x},\mathbf{y} \in \mathbb{V}$, and an operation of **scalar multiplication**, usually denoted $s\mathbf{x}$ for any $\mathbf{x} \in \mathbb{V}$ and $s \in \mathbb{C}$, such that for any $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{V}$ and $s,t \in \mathbb{C}$ we have all of the following properties:

(V1) $\mathbf{x}+\mathbf{y} \in \mathbb{V}$ (closed under addition)
(V2) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ (addition is associative)
(V3) There is an element $\mathbf{0} \in \mathbb{V}$, (called the **zero vector**), such that

$$\mathbf{x}+\mathbf{0} = \mathbf{x} = \mathbf{0}+\mathbf{x} \quad \text{(additive identity)}$$

(V4) For each $\mathbf{x} \in \mathbb{V}$, there exists an element $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$. (additive inverse)
(V5) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ (addition is commutative)
(V6) $s\mathbf{x} \in \mathbb{V}$ (closed under scalar multiplication)
(V7) $s(t\mathbf{x}) = (st)\mathbf{x}$ (scalar multiplication is associative)
(V8) $(s+t)\mathbf{x} = s\mathbf{x}+t\mathbf{x}$ (scalar addition is distributive)
(V9) $s(\mathbf{x}+\mathbf{y}) = s\mathbf{x}+s\mathbf{y}$ (scalar multiplication is distributive)
(V10) $1\mathbf{x} = \mathbf{x}$ (scalar multiplicative identity)

Matrix spaces and polynomials spaces can also be naturally extended to complex vector spaces, by simply allowing the entries or coefficients to be complex numbers. Eventually, as we did with real vector spaces, we would come to find that all finite dimensional complex vector spaces are isomorphic to some $\mathbb{C}^n$, so in this course we will focus our attention on the study of $\mathbb{C}^n$.