Solution to Practice 1e

D1(b) To prove that the zero vector is unique, we assume that two vectors (say $\mathbf{a}$ and $\mathbf{b}$) both satisfy the defining property of the zero vector. That is, let $\mathbf{a}$ and $\mathbf{b}$ be such that, for all $\mathbf{x}$

$$\begin{aligned} \mathbf{x}+\mathbf{a} = \mathbf{x} = \mathbf{a}+\mathbf{x} \quad &\text{and} \\ \mathbf{x}+\mathbf{b} = \mathbf{x} = \mathbf{b}+\mathbf{x} \quad & \end{aligned}$$

Since $\mathbf{x}+\mathbf{b} = \mathbf{x}$ for all $\mathbf{x}$, we can substitute in $\mathbf{a}$ and see that $\mathbf{a}+\mathbf{b} = \mathbf{a}$. And since $\mathbf{x} = \mathbf{a}+\mathbf{x}$ for all $\mathbf{x}$, we can substitute in $\mathbf{b}$ and see that $\mathbf{b} = \mathbf{a}+\mathbf{b}$. This means that $\mathbf{b} = \mathbf{a}+\mathbf{b} = \mathbf{a}$, so we have $\mathbf{a} = \mathbf{b}$, as desired.

D1(c) Since we know that the zero vector is unique, we can show that a vector $\mathbf{a}$ is the zero vector by showing that it satisfies the defining property of the zero vector: $\mathbf{x}+\mathbf{a} = \mathbf{x} = \mathbf{a}+\mathbf{x}$. And so, to show that $t\mathbf{0} = \mathbf{0}$, we will show that $\mathbf{x}+t\mathbf{0} = \mathbf{x} = t\mathbf{0}+\mathbf{x}$ for all $\mathbf{x}$.

First, if $t = 0$, then by Theorem 4.2.1(1) we already know that $0\mathbf{0} = \mathbf{0}$. So, let us now assume that $t \neq 0$, and let $\mathbf{x} \in \mathbb{V}$. Then

$$\begin{aligned} \mathbf{x}+t\mathbf{0} &= 1\mathbf{x}+t\mathbf{0} && \text{by V10} \\ &= ((t)(1/t))\mathbf{x}+t\mathbf{0} && \text{operation of numbers in } \mathbb{R} \\ &= t((1/t)\mathbf{x})+t\mathbf{0} && \text{by V7} \\ &= t((1/t)\mathbf{x}+\mathbf{0}) && \text{by V9} \\ &= t((1/t)\mathbf{x}) && \text{by V3} \\ &= ((t)(1/t))\mathbf{x} && \text{by V7} \\ &= 1\mathbf{x} && \text{operation of numbers in } \mathbb{R} \\ &= \mathbf{x} && \text{by V10} \end{aligned}$$

We have shown that $\mathbf{x}+t\mathbf{0} = \mathbf{x}$. And by V5, we know that $\mathbf{x}+t\mathbf{0} = t\mathbf{0}+\mathbf{x}$, so we have that

$$\mathbf{x}+t\mathbf{0} = \mathbf{x} = t\mathbf{0}+\mathbf{x}$$

and this means that $t\mathbf{0}$ is the zero vector. That is, $t\mathbf{0} = \mathbf{0}$.

D3 V1: Since the product of two positive numbers is a positive number, we know that $xy > 0$, so $x \oplus y \in \mathbb{V}$.

V2: $(x \oplus y) \oplus z = (xy)z = x(yz) = x \oplus (y \oplus z)$.

V3: We need an element $\mathbf{0}$ such that $x \oplus \mathbf{0} = x\mathbf{0} = x$. This last equality lets us know that $\mathbf{0} = 1$, and thankfully we have $1 \in \mathbb{V}$. And sure enough, since $(x)(1) = x = (1)(x)$, we have that $x \oplus 1 = x = 1 \oplus x$.

V4: Now, for every $x \in \mathbb{V}$, we need to find a $y \in \mathbb{V}$ such that $x \oplus y = 1$. That is, we want $xy = 1$. This means that $y = (1/x)$, and since $x > 0$, we know that $(1/x) > 0$. So, for each $x \in \mathbb{V}$, there is $(1/x) \in \mathbb{V}$ such that $x \oplus (1/x) = 1$.

V5: $x \oplus y = xy = yx = y \oplus x$

V6: $s \odot x = x^s > 0$, since $x > 0$, so $s \otimes x \in \mathbb{V}$.

V7: $s \odot (t \odot x) = s \odot x^t = (x^t)^s = x^{(ts)} = x^{(st)} = (st) \odot x$

V8: $(s + t) \odot x = x^{(s+t)} = x^s x^t = x^s \oplus x^t = (s \odot x) \oplus (t \odot x)$

V9: $s \odot (x \oplus y) = s \odot (xy) = (xy)^s = x^s y^s = x^s \oplus y^s = (s \odot x) \oplus (s \odot y)$

V10: $1 \odot x = x^1 = x$