Solution to Practice 1b

B1(a) $\begin{bmatrix} 3 & -2 \\ -4 & 1 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ 1 & 4 \\ -6 & -9 \end{bmatrix} = \begin{bmatrix} 3+5 & -2+4 \\ -4+1 & 1+4 \\ 3-6 & 7-9 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ -3 & 5 \\ -3 & -2 \end{bmatrix}$

B1(b) $(-5)\begin{bmatrix} 2 & 3 & -6 & -2 \\ -7 & 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -5(2) & -5(3) & -5(-6) & -5(-2) \\ -5(-7) & -5(1) & -5(0) & -5(5) \end{bmatrix} =$
$\begin{bmatrix} -10 & -15 & 30 & 10 \\ 35 & -5 & 0 & -25 \end{bmatrix}$

B1(c) $\begin{bmatrix} 4 & 2 & 3 \\ -2 & 1 & 5 \end{bmatrix} - 4\begin{bmatrix} -2 & -1 & 5 \\ 6 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 4-4(-2) & 2-4(-1) & 3-4(5) \\ -2-4(6) & 1-4(7) & 5-4(1) \end{bmatrix} =$
$\begin{bmatrix} 12 & 6 & -17 \\ -26 & -27 & 1 \end{bmatrix}$

B10 To see if $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \right\}$, we need to determine whether or not there are scalars t_1, t_2, and t_3 such that

$$t_1 \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + t_2 \begin{bmatrix} -2 & 3 \\ 0 & -1 \end{bmatrix} + t_3 \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

That is, we need to see if there are scalars t_1, t_2, and t_3 such that

$$\begin{bmatrix} t_1 - 2t_2 - t_3 & -t_1 + 3t_2 + 2t_3 \\ t_3 & 2t_1 - t_2 + t_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

By the definition of matrix equality, this is the same as looking for solutions to the following system of linear equations:

$$\begin{array}{rcrcrcr} t_1 & - & 2t_2 & - & t_3 & = & 2 \\ -t_1 & + & 3t_2 & + & 2t_3 & = & -1 \\ & & & & t_3 & = & 1 \\ 2t_1 & - & t_2 & + & t_3 & = & 1 \end{array}$$

To find the solutions of this system, we row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ -1 & 3 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ R_2 + R_1 \\ \\ R_4 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & 3 & -3 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

Since this last matrix has a bad row, we know that the system does not have any solutions. Therefore, $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ is NOT in the span of $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \right\}$.

B11 To see if $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 3 & -3 \end{bmatrix} \right\}$ is linearly dependent or linearly independent, we need to look at the solutions to the equation

$$t_1 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix} + t_3 \begin{bmatrix} 2 & -3 \\ 0 & -1 \end{bmatrix} + t_4 \begin{bmatrix} -3 & -1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which is the same as

$$\begin{bmatrix} t_1 - 3t_2 + 2t_3 - 3t_4 & -3t_3 - t_4 \\ -t_1 + 4t_2 + 3t_4 & -2t_2 - t_3 - 3t_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

And this is the same as the homogeneous linear system

$$\begin{array}{rcrcrcrcl} t_1 & - & 3t_2 & + & 2t_3 & - & 3t_4 & = & 0 \\ & & & - & 3t_3 & - & t_4 & = & 0 \\ -t_1 & + & 4t_2 & & & + & 3t_4 & = & 0 \\ & - & 2t_2 & - & t_3 & - & 3t_4 & = & 0 \end{array}$$

To find out how many solutions this system has, we will row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -3 & 2 & -3 \\ 0 & 0 & -3 & -1 \\ -1 & 4 & 0 & 3 \\ 0 & -2 & -1 & -3 \end{bmatrix} R_3 + R_1 \sim \begin{bmatrix} 1 & -3 & 2 & -3 \\ 0 & 0 & -3 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -1 & -3 \end{bmatrix} R_2 \updownarrow R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & -2 & -1 & -3 \end{bmatrix} R_4 + 2R_2 \sim \begin{bmatrix} 1 & -3 & 2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 3 & -3 \end{bmatrix} R_4 + R_3$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

This final matrix is in row echelon form, and so we see that the rank of the coefficient matrix is 4. As this is the same as the number of variables in the system, the general solution does not have any parameters. And therefore, we see that $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 3 & -3 \end{bmatrix} \right\}$ is linearly independent.