Solution to Practice 1a

A1(a) $\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 3+6 \\ 2-2 \\ -1+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \\ 1 \end{bmatrix}$

A1(b) $\begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3+6 \\ -2-3-2 \\ 5-3+8 \\ 1-6+0 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 10 \\ -5 \end{bmatrix}$

A1(c) $2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+4-6 \\ 4-4+0 \\ 2+2-3 \\ 0+4-3 \\ -2+2+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

B1(a) We are trying to find x_1, x_2, and x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 2 \\ -6 \end{bmatrix}$$

This is the same as a system of four equations in three unknowns, with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & -6 \end{array}\right]$$

To solve this, we will row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & -6 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ -R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From our final RREF matrix, we see that $x_1 = 2$, $x_2 = -3$, and $x_3 = -2$ is a solution to our system. As such, we have that

$$\begin{bmatrix} -4 \\ -2 \\ 2 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

B1(b) We are trying to find x_1, x_2, and x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

This is the same as a system of four equations in three unknowns, with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right]$$

To solve this, we will row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & -2 & 2 & -6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ -R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

At this point, our last row is a bad row, so we know that the system is inconsistent. That is, we know that we will not be able to find x_1, x_2 and x_3 to satisfy our equations. Thus, $\begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}$ is not in Span$B$.

B1(c) We are trying to find x_1, x_2, and x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

This is the same as a system of four equations in three unknowns, with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & 1 \end{array}\right]$$

To solve this, we will row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -2 & 2 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ -R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From our final RREF matrix, we see that $x_1 = 3$, $x_2 = 0$, and $x_3 = -2$ is a solution to our system. As such, we have that

$$\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

B5(a) To determine whether the given set is linearly independent, we need to look at the solution to the following homogeneous linear system:

$$\begin{array}{ccccccc} t_1 & + & t_2 & & & = & 0 \\ & & 2t_2 & + & t_3 & = & 0 \\ t_1 & & & + & 2t_3 & = & 0 \\ & & t_2 & + & 3t_3 & = & 0 \end{array}$$

And in order to find the solution, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \\ R_2 \updownarrow R_4 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & -5 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in row echelon form, and by looking at it we see that the rank of the coefficient matrix is 3. As this is the same as the number of vectors, by Lemma 3 our set is linearly independent.

B5(b) To determine whether the given set is linearly independent, we need to look at the solution to the following homogeneous linear system:

$$\begin{matrix} t_1 & + & t_2 & & & + & t_4 & = & 0 \\ & & 2t_2 & + & t_3 & - & 3t_4 & = & 0 \\ t_1 & & & + & t_3 & + & t_4 & = & 0 \\ t_1 & & & + & 2t_3 & & & = & 0 \\ & & t_2 & - & 2t_3 & + & t_4 & = & 0 \end{matrix}$$

And in order to find the solution, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_1 \\ R_4 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \updownarrow R_5 \\ \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 1 & -3 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \\ R_5 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ R_5 + 5R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in row echelon form, and by looking at it we see that the rank of the coefficient matrix is 3. Since this is less than the number of vectors, we know that our set is linearly dependent. As such, we must now find all linear combinations of the vectors that make $\vec{0}$. But that simply means completing the steps to find the solution to our system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -R_3 \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 + 2R_3$$
$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Putting the RREF matrix into equations, we have

$$\begin{array}{ccccccc} t_1 & & & + & 2t_4 & = & 0 \\ & t_2 & & - & t_4 & = & 0 \\ & & t_3 & - & t_4 & = & 0 \end{array}$$

and replacing the variable t_4 with the parameter t, we have

$$\begin{array}{ccccccc} t_1 & & & + & 2t & = & 0 \\ & t_2 & & - & t & = & 0 \\ & & t_3 & - & t & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \\ t \end{bmatrix}$$

As such, we see that

$$-2t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - t \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

is the list of all linear combinations of the vectors that are $\vec{0}$.