Lecture 1b

A Review of Matrices

In our attempts to solve spanning and linear independence problems in $\mathbb{R}^n$, we created a new object known as a matrix. We went on to study many properties of matrices, but for now, let's review the properties of a matrix that paralleled our definitions of $\mathbb{R}^n$. We should, of course, start with the definition of a matrix:

<u>Definition</u>: A **matrix** is a rectangular array of numbers. We say that A is an $m \times n$ matrix when A has m rows and n columns, such as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

<u>Definition</u>: Two matrices A and B are **equal** if and only if they have the same size (that is, the same number of rows and the same number of columns) and their corresponding entries are equal. That is, if $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

<u>Notation</u>: We sometimes denote the ij-th entry of a matrix A by $(A)_{ij}$. This is taken to be the same thing as a_{ij}.

And next comes the definitions of addition and scalar multiplication:

<u>Definition</u>: Let A and B be $m \times n$ matrices. We define **addition** of matrices by

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$$

That is, the ij-th entry of $A + B$ is the sum of the ij-th entry of A with the ij-th entry of B.

<u>Definition</u>: Let A be an $m \times n$ matrix, and $t \in \mathbb{R}$ a scalar. We define the **scalar multiplication** of matrices by

$$(tA)_{ij} = t(A)_{ij}$$

That is, the ij-th entry of tA is t times the ij-th entry of A.

With these definitions in hand, we notice that we get the EXACT SAME theorem of useful properties as we did for $\mathbb{R}^n$.

<u>Theorem 3.1.1</u> Let A, B, and C be $m \times n$ matrices and let s and t be real scalars. Then

(1) $A + B$ is an $m \times n$ matrix (closed under addition)
(2) $A + B = B + A$ (addition is commutative)
(3) $(A + B) + C = A + (B + C)$ (addition is associative)
(4) There exists a matrix, denoted by $O_{m,n}$, such that $A + O_{m,n} = A$ (zero matrix)
(5) For each matrix A, there exists an $m \times n$ matrix $(-A)$, with the property that $A + (-A) = O_{m,n}$ (additive inverses)
(6) sA is an $m \times n$ matrix (closed under scalar multiplication)
(7) $s(tA) = (st)A$ (scalar multiplication is associative)
(8) $(s + t)A = sA + tA$ (distributive law)
(9) $s(A + B) = sA + sB$ (distributive law)
(10) $1A = A$ (scalar multiplicative identity)

We were also able to define spanning sets and linear independence in matrices, much as we had done in $\mathbb{R}^n$.

<u>Definition</u>: Let $\mathcal{B} = \{A_1, \ldots, A_k\}$ be a set of $m \times n$ matrices. Then the **span** of $\mathcal{B}$ is defined as

$$\text{Span}\mathcal{B} = \{t_1A_1 + \cdots + t_kA_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$$

That is, Span$\mathcal{B}$ is the set of all linear combinations of the matrices in $\mathcal{B}$.

Example: Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

To do this, we need to see if there are scalars t_1, t_2, t_3, and t_4 such that

$$t_1 \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + t_3 \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} + t_4 \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Performing the operation on the left side, we see that we need

$$\begin{bmatrix} t_1 - 2t_3 + 2t_4 & t_1 + 3t_2 + 4t_3 + 2t_4 \\ 2t_1 + 4t_2 - 4t_3 - 4t_4 & 2t_1 - 3t_2 - 5t_3 + 3t_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

By the definition of equality, this means we are looking for solutions to the following system of linear equations:

$$\begin{array}{rcrcrcrcr} t_1 & & & - & 2t_3 & + & 2t_4 & = & -1 \\ t_1 & + & 3t_2 & + & 4t_3 & + & 2t_4 & = & 2 \\ 2t_1 & + & 4t_2 & - & 4t_3 & - & 4t_4 & = & 2 \\ 2t_1 & - & 3t_2 & - & 5t_3 & + & 3t_4 & = & 1 \end{array}$$

We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 1 & 3 & 4 & 2 & 2 \\ 2 & 4 & -4 & -4 & 2 \\ 2 & -3 & -5 & 3 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 3 & 6 & 0 & 3 \\ 0 & 4 & 0 & -8 & 4 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{c} (1/3)R_2 \\ (1/4)R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 + 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] (-1/2)R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] R_4 - 5R_3 \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -6 & 6 \end{array}\right] (-1/6)R_4$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 - 2R_4 \\ \\ R_3 - R_4 \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 + 2R_3 \\ R_2 - 2R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

We see from the RREF matrix that $t_1 = 3$, $t_2 = -1$, $t_3 = 1$, $t_4 = -1$ is a solution to our system. This means that

$$3\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

and thus, that $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ IS in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Definition: Let $\mathcal{B} = \{A_1, \ldots, A_k\}$ be a set of $m \times n$ matrices. Then $\mathcal{B}$ is said to be **linearly independent** if the only solution to the equation

$$t_1 A_1 + \cdots + t_k A_k = O_{m,n}$$

is the trivial solution $t_1 = \cdots = t_k = 0$. Otherwise, $\mathcal{B}$ is said to be **linearly dependent**.

Example: Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ is linearly independent.

To do this, we need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} + t_2 \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix} + t_3 \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix} t_1 + 8t_3 & 3t_1 - 2t_2 + 6t_3 \\ -t_1 + t_2 + 5t_3 & -3t_1 + 5t_2 + t_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and this is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & & & + & 8t_3 & = & 0 \\ 3t_1 & - & 2t_2 & + & 6t_3 & = & 0 \\ -t_1 & + & t_2 & + & 5t_3 & = & 0 \\ -3t_1 & + & 5t_2 & + & t_3 & = & 0 \end{array}$$

We solve this system by row reducing the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 8 \\ 3 & -2 & 6 \\ -1 & 1 & 5 \\ -3 & 5 & 1 \end{bmatrix} \begin{array}{c} \\ R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & -2 & 18 \\ 0 & 1 & 13 \\ 0 & 5 & 25 \end{bmatrix} \begin{array}{c} \\ (-1/2)R_2 \\ \\ (1/5)R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 1 & 13 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 14 \end{bmatrix} \begin{array}{c} \\ \\ \\ R_4 - (14/22)R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 0 \end{bmatrix}.$$

This final matrix is in row echelon form, and so we see that the rank of the coefficient matrix is 3. Since this is the same as the number of variables, there are no parameters in the general solution to our homogeneous system. This means that there is only one solution to the system, and we know that this must be the trivial solution. And this means that the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ IS linearly independent.

Example: Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} \right\}$ is linearly independent.

To do this, we need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} + t_2 \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix} + t_3 \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix} t_1 - t_2 + 2t_3 & t_1 + 2t_2 + 11t_3 \\ 3t_1 - 8t_2 - 9t_3 & -t_1 + 3t_2 + 4t_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and this is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & - & t_2 & + & 2t_3 & = & 0 \\ t_1 & + & 2t_2 & + & 11t_3 & = & 0 \\ 3t_1 & - & 8t_2 & - & 9t_3 & = & 0 \\ -t_1 & + & 3t_2 & + & 4t_3 & = & 0 \end{array}$$

We solve this system by row reducing the coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 11 \\ 3 & -8 & -9 \\ -1 & 3 & 4 \end{bmatrix} \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 + R_1 \end{array} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 9 \\ 0 & -5 & -15 \\ 0 & 2 & 6 \end{bmatrix} \begin{array}{c} \\ (1/3)R_2 \\ (-1/5)R_3 \\ (1/2)R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - R_3 \end{array} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This final matrix is in row echelon form, so we see that the rank of the coefficient matrix is 2. Since the number of variables in the system is 3, this means that there are 3-2=1 parameters in the general solution to the system. Thus, $t_1 = t_2 = t_3 = 0$ is not the only solution to our equation, and this means that $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} \right\}$ is linearly dependent. (That is, it is NOT linearly independent.)