Lecture 1a

A Review of Vectors in $\mathbb{R}^n$

We began our studies of linear algebra by looking at collections of numbers. Let's take a moment to review the definition and properties of $\mathbb{R}^n$.

Definition $\mathbb{R}^n$ is the set of all vectors of the form $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, where $x_i \in \mathbb{R}$. That is

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} | x_1, \ldots, x_n \in \mathbb{R} \right\}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then we define addition of vectors componentwise by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and we define scalar multiplication componentwise by

$$t\vec{x} = t \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix}$$

Once we defined $\mathbb{R}^n$, we noted the following properties of vector addition and scalar multiplication.

Theorem 1.2.1: For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have
(1) $\vec{x} + \vec{y} \in \mathbb{R}^n$ (closed under addition)
(2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (addition is commutative)
(3) $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ (addition is associative)
(4) There exists a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{z} + \vec{0} = \vec{0}$ for all $\vec{z} \in \mathbb{R}^n$ (zero vector)

(5) For each $\vec{x} \in \mathbb{R}^n$ there exists a vector $-\vec{x} \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverse)
(6) $t\vec{x} \in \mathbb{R}^n$ (closed under scalar multiplication)
(7) $s(t\vec{x}) = (st)\vec{x}$ (scalar multiplication is associative)
(8) $(s+t)\vec{x} = s\vec{x} + t\vec{x}$ (a distributive law)
(9) $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$ (another distributive law)
(10) $1\vec{x} = \vec{x}$ (scalar multiplicative identity)

After this, we defined two recurring concepts: spanning sets and linear independence.

Definition If S is the subspace of $\mathbb{R}^n$ consisting of all possible linear combinations of the vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, then S is called the subspace **spanned** by the set of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$, and we say that the set $\mathcal{B}$ **spans** S. The set $\mathcal{B}$ is called a **spanning set** for the subspace S. We denote S by

$$S = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{Span}\mathcal{B}$$

Example: Determine whether or not $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} \right\}$.

To solve this we need to determine whether or not there are any solutions to the vector equation $t_1 \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix} + t_2 \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. Breaking this vector equation into its three components, we see that the vector equation has a solution if and only if the following system of linear equations has a solution:

$$\begin{array}{rcrcr} -3t_1 & + & 2t_2 & = & 1 \\ 2t_1 & - & 4t_2 & = & 2 \\ 6t_3 & - & 13t_2 & = & 4 \end{array}$$

To determine whether or not this system has a solution, we will write it as an augmented matrix and row reduce:

$$\left[\begin{array}{cc|c} -3 & 8 & 1 \\ 2 & -4 & 2 \\ 6 & -13 & 4 \end{array}\right] \; (1/2)R_2 \sim \left[\begin{array}{cc|c} -3 & 8 & 1 \\ 1 & -2 & 1 \\ 6 & -13 & 4 \end{array}\right] \; R_1 \updownarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ -3 & 8 & 1 \\ 6 & -13 & 4 \end{array}\right] \; \begin{array}{l} R_2 + 3R_1 \\ R_3 - 6R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{array}\right] \; (1/2)R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{array}\right] \; R_3 + R_2 \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The final matrix is in row echelon form, and as it has no bad rows, we know that the system is consistent. This means that $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ IS in Span $\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} \right\}$.

Definition A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be **linearly dependent** if there exists coefficients $t_1, \ldots, t_k$ not all zero such that

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be **linearly independent** if the only solution to

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

is $t_1 = \cdots = t_k = 0$. This is called the **trivial solution**.

Example: Determine whether the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

To do this, we need to see if there are any parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} t_1 & + & t_2 & - & 3t_3 & = & 0 \\ 2t_1 & + & 4t_2 & - & 4t_3 & = & 0 \\ -t_1 & + & 7t_2 & & & = & 0 \end{array}$$

To determine this, we will row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 4 & -4 \\ -1 & 7 & 0 \end{bmatrix} \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 2 \\ 0 & 8 & -3 \end{bmatrix} \begin{array}{c} \\ (1/2)R_2 \\ \\ \end{array}$$
$$\sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 8 & -3 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - 8R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -11 \end{bmatrix}$$

This last matrix is in row echelon form, and thus we see that the rank of the coefficient matrix is 3. Since this is the same as the number of variables (which is the same as the number of vectors), there are no parameters in the general solution. And this means that the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ IS linearly independent.

Example: Determine whether the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

Again, this will come down to determining whether or not there are parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} & & t_2 & + & 3t_3 & = & 0 \\ 3t_1 & - & 7t_2 & & & = & 0 \\ -2t_1 & + & 6t_2 & + & 4t_3 & = & 0 \end{array}$$

To determine this, we will row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ -2 & 6 & 4 \end{bmatrix} \; (-1/2)R_3 \quad \sim \begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ 1 & -3 & -2 \end{bmatrix} \; R_1 \updownarrow R_3$$

$$\sim \begin{bmatrix} 1 & -3 & -2 \\ 3 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix} \; R_2 - 3R_1 \; \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix} \; R_3 - (1/2)R_2$$

$$\sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in row echelon form, and thus we see that the rank of the coefficient matrix is 2. This means that there is one parameter in the general solution to the homogeneous system. And this means that the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ in NOT linearly independent. (That is, our set is linearly dependent.)