Solution to Practice 3u

B2(a) $\left[\begin{array}{rrr|rrr} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_3 \\ \sim \\ \end{array} \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 \end{array}\right] R_3 - 2R_1$

$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & -2 \end{array}\right] R_3 - R_2 \sim \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & -1 \end{array}\right] (1/2)R_3$

$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ R_2 - R_3 \end{array} \sim \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 3/2 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1 \end{array}\right] R_1 + R_2$

$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1/2 & 3/2 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1 \end{array}\right].$

So we see that $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -1/2 & 3/2 & 1 \\ 1/2 & -1/2 & -1 \end{bmatrix}$

B2(b)(i) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 0 & 1 & 1 \\ -1/2 & 3/2 & 1 \\ 1/2 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0+3-1 \\ -1+9/2-1 \\ 1-3/2+1 \end{bmatrix} =$ $\begin{bmatrix} 2 \\ 5/2 \\ 1/2 \end{bmatrix}$. We verify this by noting that $A \begin{bmatrix} 2 \\ 5/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

B2(b)(ii) It would be easy enough to calculate $3 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix}$, and then get that $\vec{x} = A^{-1} \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix}$. But we can also solve this using the linearity properties of matrix multiplication and our result from (i). For, if $\vec{x}$ is such that $A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, then $A(3\vec{x}) = 3(A\vec{x}) = 3 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Thus, we see that the solution to $A\vec{x} = 3 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is $3 \begin{bmatrix} 2 \\ 5/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 6 \\ 15/2 \\ 3/2 \end{bmatrix}$.

B2(b)(iii) $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 0 & 1 & 1 \\ -1/2 & 3/2 & 1 \\ 1/2 & -1/2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-2+3 \\ -2-3+3 \\ 2+1-3 \end{bmatrix} =$ $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, which we can verify by noting that $A \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$.

D5 Let A be an $n \times n$ matrix.

if (1), then (2): Suppose A is invertible. Then the rank of A is n. By the rank-nullity theorem, we get that the nullity of A is zero. Thus, $\text{Null}(A) = \{\vec{0}\}$.

if (2), then (3): If $\text{Null}(A) = \{\vec{0}\}$, then the nullity of A is zero, and thus by the rank-nullity theorem, the rank of A is n. This means that the reduced row echelon form of A is I and thus that $\text{Row}(A) = \text{Row}(I) = \mathbb{R}^n$. If we let $\vec{a}_1^T, \ldots, \vec{a}_n^T$ be the rows of A, then we have that $\{\vec{a}_1, \ldots, \vec{a}_n\}$ spans $\mathbb{R}^n$. By Lemma 1 on p. 94, this means that the rank of the coefficient matrix of the system $t_1\vec{a}_1 + \cdots + t_n\vec{a}_n = \vec{v}$ is n. (Note: Said coefficient matrix is A^T, but that won't actually be important for this proof.) And thus, by Theorem 5 on p.98, we see that $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is a basis for $\mathbb{R}^n$. By the definition of "basis", we get that the set $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is linearly independent. That is, we see that the rows of A are linearly independent.

if (3), then (4): Suppose the rows of A are linearly independent. Since the rows of A are the columns of A^T, this means that the columns of A^T are linearly independent. And thus A^T is invertible (by the Invertible Matrix Theorem).

if (4), then (1): Suppose A^T is invertible. Then, using Theorem 3 on p. 166, we see that $((A^T)^{-1})^T = ((A^T)^T)^{-1} = A^{-1}$. And so we have that A is invertible.

Note: I have included specific references to theorems to aid the reader in remembering the various theorems presented in the course. As the student, you need not include these references on quizzes or exams. I'll know if you're statements are true or not!