Lecture 3w

Elementary Matrices

(pages 176-7)

Elementary row operations have become very important to us in this course, but just as we did with matrices, we can not help but wonder if there isn't even more we can do with them. Since our other favourite subject is linearity, perhaps it is time to wonder: are elementary operations a linear mapping? Well, to start, we would have to change our definition to allow inputs other than vectors, but the basic properties of linearity are that you preserve addition and scalar multiplication. Consider the following example:

Example: Let L be the function whose domain and codomain are the set of 2×2 functions, and let L be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$$

That is, suppose L maps a matrix to the row equivalent matrix obtained by adding two times the second row to the first row. Then consider the following:

$$\begin{aligned} L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1+a_2+2c_1+2c_2 & b_1+b_2+2d_1+2d_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1+2c_1 & b_1+2d_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2+2c_2 & b_2+2d_2 \\ c_2 & d_2 \end{bmatrix} \\ &= L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \end{aligned}$$

$$\begin{aligned} L\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= L\left(\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}\right) \\ &= \begin{bmatrix} sa+2sc & sb+2sd \\ sc & sd \end{bmatrix} \\ &= s\begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} \\ &= sL\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \end{aligned}$$

So we see that the function satisfies the linearity properties of preserving addition and scalar multiplication.

Now, the purpose of this lecture is not to expand the study of linear mappings to sets of matrices. (People do, though. Math 225 will explore this and other generalizations of linear mappings.) Instead, I was hoping to inspire the idea that if row operations are linear, then there should be a standard matrix for them. And in fact, it turns out that there is such a matrix, and even better, that it is very easy to find.

Definition: A matrix that can be obtained from the identity matrix by a single elementary row operation is called an **elementary matrix**.

Okay, that definition may not seem like it was where the discussion was heading, but it turns out that an elementary matrix is the standard matrix for the row operation used to create it from the identity matrix. Let's look at some examples to see what I mean.

Example: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is an elementary matrix, because it can be obtained from I_2 by adding 2 times the second row to the first row. But notice that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$$

And so, if we multiply a 2×2 matrix ON THE LEFT by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, the resulting matrix is the same as if we had added two times the second row of our matrix to the first row. Which is the elementary row operation used to get $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ from I_2.

Example: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is an elementary matrix, because it is obtained from I_3 by multiplying the third row by 5. But notice that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

So multiplying ON THE LEFT by E maps a matrix to the row equivalent matrix you get by multiplying the third row by 5. Just as E was obtained from I_3 by multiplying the third row by 5.

Example: $F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix, because it is the matrix you get when you switch the second and third rows of I_4. And we also see that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \\ c & d \\ g & h \end{bmatrix}$$

So multiplying by F ON THE LEFT has the same effect as the elementary row operation of switching the second and third rows. Just as F was obtained from I_4 by switching the second and third rows.

You'll have noticed that I keep emphasizing that you multiply the elementary matrix ON THE LEFT. Because matrix multiplication is not commutative, you will not achieve the appropriate row operation if you multiply the elementary matrix on the right.

Example: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$, which is not what you get if you add 2 times the second row of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to its first row.

Example: $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not an elementary matrix, because you need two row operations to get A from I_3, namely $2R_1$, followed by $R_1 + R_3$. Remember that replacing R_1 with $2R_1 + R_3$ is not an elementary row operation. Replacing R_1 with $R_1 + 2R_3$ IS an elementary row operation, which has matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in $\mathbb{R}^3$.

On the strength of these examples (because a full proof would be tedious), we summarize our findings in the following theorem:

Theorem 3.6.1: If A is an $n \times n$ matrix and E is the elementary matrix obtained from I_n by a certain elementary row operatioin, then the product EA is the matrix obtained from A by performing the same elementary row operation.

Multiple uses of Theorem 3.6.1 gets us our next result.

Theorem 3.6.2: For any $m \times n$ matrix A, there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \ldots E_2 E_1 A$ is equal to the reduced row echelon form of A.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 7 \\ -1 & -1 \end{bmatrix}$. Find a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \ldots E_2 E_1 A$ is the reduced row echelon form of A.

To do this, we first need to row reduce A to its reduced row echelon form. But instead of caring what the reduced row echelon form is, we only care what row operations we use to get there.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 9 \\ -1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 3 \\ 0 & 3 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} R_2 \updownarrow R_3$$
$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} (1/3)R_2 \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The first elementary row operation is... Oh wait, I did three at the same time. Does this mean that I should combine them into one matrix? NO–such a matrix would not be an elementary matrix. But back when I set up the rules for when you can do multiple row operations "at the same time", I had this day in mind. Because it doesn't matter which order these row operations take place–you always end up with the same matrix after you do all three of them. The reality is that there are lots of different ways to row reduce a matrix, so although the reduced row echelon form of a matrix is unique, the sequence $E_1, \ldots, E_k$ will not be unique. As such, with these questions, it is especially important to write down your row operations, as that is the basis upon which your choices of $E_1, \ldots, E_k$ will be judged. And now, back to our example:

The first elementary row operation is $R_2 - 2R_1$, so $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The second elementary row operation is $R_3 - 3R_1$, so $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The third elementary row operation is $R_4 + R_1$, so $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

The fourth elementary row operation is $R_4 - R_3$, so $E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.

The fifth elementary row operation is $R_2 \updownarrow R_3$, so $E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The sixth elementary row operation is $(1/3)R_2$, so $E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The seventh elementary row operation is $R_1 - 2R_2$, so $E_7 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.