Lecture 3u

Properties of Invertible Matrices

(pages 168-9)

Through our study of invertible matrices, we discovered that an $n \times n$ matrix A is invertible if and only if the rank of A is n. This means that we can add "A is invertible" to the list of things we know about a matrix with rank n. In fact, there are a lot of things we know about an invertible matrix.

Theorem 3.5.4 (Invertible Matrix Theorem): Suppose that A is an $n \times n$ matrix. Then the following statements are equivalent. (That is, one is true if and only if they are all true.)

(1) A is invertible.
(2) $\text{rank}(A) = n$
(3) The reduced row echelon form of A is I.
(4) For all $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ is consistent and has a unique solution.
(5) The columns of A are linearly independent.
(6) The columnspace of A is $\mathbb{R}^n$.

Proof of Theorem 3.5.4: Theorem 3.5.4 is reasonably straightforward to prove, but it gives us a chance to look at the structure of TFAE ("The Following Are Equivalent") proofs. The idea is that instead of actually proving the if and only if between each pair, we set up a circular chain of "if-then" statements: if (1), then (2); if (2), then (3); if (3), then (4); if (4), then (5); if (5), then (6); and lastly, if (6), then (1), creating our loop. So, if what we are really interested in is "if (5), then (2)", then we combine the proofs of "if (5), then (6)", "if (6), then (1)", and "if (1), then (2)". So we can start from anywhere in our loop, and end up anywhere else! Of course, we still need to prove our 6 if-then statements:

if (1), then (2): From Theorem 3.5.2

if (2), then (3): This follows from the definition of rank and the fact that A is an $n \times n$ matrix.

if (3), then (4): Suppose that the reduced row echelon form of A is I, and consider how we would look for a solution to $A\vec{x} = \vec{b}$. We would row reduce the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$. Since the reduced row echelon form of A is I, we would end up with $\left[\begin{array}{c|c} I & \vec{y} \end{array}\right]$. This is the structure of a consistent system with a unique solution (namely $\vec{y}$). And since this works for all $\vec{b} \in \mathbb{R}^n$, we see that for all $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ is consistent and has a unique solution.

if (4), then (5): Suppose that for all $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ is consistent and has a unique solution. Then we specifically know that the system $A\vec{x} = \vec{0}$ has a unique solution. If we let the columns of A be $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, then the system

$A\vec{x} = \vec{0}$ is the same as the vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$, and so we know that our vector equation has only one solution. Thus, the vectors are linearly independent. That is, we see that the columns of A are linearly independent.

if (5), then (6): Suppose that the columns of A are linearly independent, and let $\vec{a}_1, \dots, \vec{a}_n$ be the columns of A. Then $A\vec{x} = \vec{0}$ has a unique solution, and therefore the rank of A must be n. And since A is the coefficient matrix of $t_1\vec{a}_1 + \cdots + t_n\vec{a}_n = \vec{v}$, and since A has rank n, we see that $\{\vec{a}_1, \dots, \vec{a}_n\}$ is a basis for $\mathbb{R}^n$. As such, the span of $\{\vec{a}_1, \dots, \vec{a}_n\}$ is $\mathbb{R}^n$. That is, we see that the span of the columns of A (i.e. the columnspace of A) is $\mathbb{R}^n$.

if (6), then (1): Suppose that the columnspace of A is $\mathbb{R}^n$. Then for every $\vec{z} \in \mathbb{R}^n$, there is some $\vec{y} \in \mathbb{R}^n$ such that $A\vec{y} = \vec{z}$. And thus, we know that for every standard basis vector $\vec{e}_i$, there is some vector $\vec{y}_i$ such that $A\vec{y}_i = \vec{e}_i$. Let B be the matrix whose columns are the $\vec{y}_i$. (That is, let $B = \begin{bmatrix} \vec{y}_1 & \vec{y}_2 & \cdots & \vec{y}_n \end{bmatrix}$.) Then $AB = \begin{bmatrix} A\vec{y}_1 & A\vec{y}_2 & \cdots & A\vec{y}_n \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} = I$. Thus, $B = A^{-1}$, and so A is invertible.

The Invertible Matrix Theorem tells us that if A is invertible, then the system $A\vec{x} = \vec{b}$ is consistent, with a unique solution. In fact, the situation is even better, because we get that $A^{-1}\vec{b} = A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} = I\vec{x} = \vec{x}$. Summarizing, we have that the solution to $A\vec{x} = \vec{b}$ is $A^{-1}\vec{b}$.

Example: Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$. In the previous lecture we found that $A^{-1} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$. Thus, we see that the solution to $A\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is $A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10+4 \\ 8-3 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$. We can verify this result by checking that $A\begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -18+20 \\ -24+25 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, as desired.