Lecture 3t

Finding the Inverse

(pages 167-8)

Let A be an $n \times n$ matrix. Then the search for A^{-1} is the search for an unknown matrix X such that $AX = I$. If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are the columns of X, then our old friend block multiplication turns the equation $AX = I$ into

$$\left[\begin{array}{cccc} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{array} \right] = \left[\begin{array}{cccc} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{array} \right]$$

And since two matrices are equal if and only if their columns are equal, this matrix equation is the same as the following list of systems:

$$A\vec{x}_1 = \vec{e}_1, \qquad A\vec{x}_2 = \vec{e}_2, \cdots, A\vec{x}_n = \vec{e}_n$$

In general, to find $\vec{x}_i$ such that $A\vec{x}_i = \vec{e}_i$, we would row reduce the augmented matrix $[A|\vec{e}_i]$. But, since all these systems have the same coefficient matrix (A), we would use the exact same row reduction steps to solve them all (namely, the ones we would use to get A into reduced row echelon form). And since row operations take place within a column, we can actually solve all these systems simultaneously by looking at the following multi-augmented matrix:

$$\left[\begin{array}{c|c|c|c|c} A & \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{array} \right]$$

If A is invertible, then Theorem 3.5.2 tells us that the rank of A is n, and thus the reduced row echelon form of A is I. So, *if* A is invertible, when we row reduce we will end up with

$$\left[\begin{array}{c|c|c|c|c} I & \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right]$$

Where the vectors $\vec{x}_i$ are our solutions to $A\vec{x}_i = \vec{e}_i$, and thus are the columns of $X = A^{-1}$.

In general, when looking for the inverse of A, we drop the extra augmentation lines, and simply look at row reducing the matrix $[A|I]$. In the case when A is invertible, we will end up with $[I|A^{-1}]$. Also worth noting at this point is that if A has rank n, then $[A|I]$ will row reduce to $[I|B]$, and by the same arguments as above we know that $B = A^{-1}$, so now see that if A has rank n, then A is invertible. (Before we only knew that if A was invertible, then A has rank n. Now we have an "if and only if" statement.)

But what if A isn't invertible? Then at least one of our systems $A\vec{x}_i = \vec{e}_i$ does not have a solution. What would that look like? Well, when we row reduce $[A|\vec{e}_i]$, we would end up with a bad row. And that means that the reduced row echelon form of A has at least one row of zeros. Which makes sense, seeing as we've already seen that if A is not invertible, then it does not have rank n. So, to determine whether or not A is invertible, we need to see whether or not A has rank n, or equivalently to see whether or not the reduced row echelon form of A is I.

These results are summarized in the following algorithm:

Algorithm: To find the inverse of a square matrix A,

(1) Row reduce the multi-augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$ so that the left block is in reduced row echelon form.

(2) If the reduced row echelon form is $\left[\begin{array}{c|c} I & B \end{array}\right]$, then $A^{-1} = B$.

(3) If the reduced row echelon form of A is not I, then A is not invertible.

Example: To determine if $A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ is invertible, we will row reduce the matrix $\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$ as follows:

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \begin{array}{l} (1/3)R_1 \\ \end{array} \sim \left[\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 4R_1 \end{array}$$
$$\sim \left[\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 0 & -1/3 & -4/3 & 1 \end{array}\right] \begin{array}{l} \\ -3R_2 \end{array} \sim \left[\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 0 & 1 & 4 & -3 \end{array}\right] \begin{array}{l} R_1 - (4/3)R_2 \\ \end{array}$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 4 \\ 0 & 1 & 4 & -3 \end{array}\right].$$

From this, we see that $A^{-1} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$ We can (and should!) verify this by looking at $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} (3)(-5)+(4)(4) & (3)(4)+(4)(-3) \\ (4)(-5)+(5)(4) & (4)(4)+(5)(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example: To determine if $B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 8 \\ 2 & 1 & 1 & -5 \\ 2 & 2 & 8 & 6 \end{bmatrix}$ is invertible, we need to row reduce $\left[\begin{array}{c|c} B & I \end{array}\right]$ as follows:

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 8 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & -5 & 0 & 0 & 1 & 0 \\ 2 & 2 & 8 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -5 & -2 & 0 & 1 & 0 \\ 0 & -2 & 6 & 6 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (1/2)R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 1/2 & 1/2 & 0 & 0 \\ 0 & -3 & -1 & -5 & -2 & 0 & 1 & 0 \\ 0 & -2 & 6 & 6 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 3R_2 \\ R_4 + 2R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 5 & 7 & -1/2 & 3/2 & 1 & 0 \\ 0 & 0 & 10 & 14 & -1 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 2R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 5 & 7 & -1/2 & 3/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & 1 \end{array}\right]$$

We need not continue the process of row reducing further, because we have a row of all zeros in the left side, which means that B is not row equivalent to I, and thus B is not invertible.