Solution to Practice 3r

B8(a) The rows of A have 7 entries, so Row(A) is a subspace of $\mathbb{R}^7$.

B8(b) The basis for Row(A) is the non-zero rows of the RREF:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$$

This is based on the fact that elementary row operations do not change the row space, so A and R have the same row space. And the structure of a RREF matrix makes it easy to see that its non-zero rows are linearly independent.

B8(c) The columns of A have 5 entries, so Col(A) is a subspace of $\mathbb{R}^5$.

B8(d) The columns of A that correspond to the columns of R with a leading 1 form a basis for Col(A):

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

This is because the RREF shows us that these columns are linearly independent. We can not simply use the columns of R, however, as elementary row operations can change the column space.

B8(e) Thinking of R as the RREF of the coefficient matrix for $A\vec{x} = \vec{0}$, we know that $A\vec{x} = \vec{0}$ is equivalent to the system:

$$\begin{array}{ccccccccc} x_1 & +2x_2 & & & +3x_5 & & -x_7 & = 0 \\ & & x_3 & & -x_5 & & -2x_7 & = 0 \\ & & & x_4 & -2x_5 & & +x_7 & = 0 \\ & & & & & x_6 & +4x_7 & = 0 \end{array}$$

Replacing the variables x_2, x_5, and x_7 with the parameters r, s, and t, we get

$$\begin{array}{ccccccccc} x_1 & +2r & & & +3s & & -t & = 0 \\ & & x_3 & & -s & & -2t & = 0 \\ & & & x_4 & -2s & & +t & = 0 \\ & & & & & x_6 & +4t & = 0 \end{array}$$

And now we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} -2r-3s+t \\ r \\ s+2t \\ 2s-t \\ s \\ -4t \\ t \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

So, a spanning set for the solution space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$

B8(f) If we look at the entries corresponding to the variable we turned into parameters (i.e. the 2nd, 5th, and 7th entries) we see that exactly one of the vectors has a 1 in this entry, and the other vectors have a 0 in this entry. As such, when we look for solutions to

$$t_1\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 0 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the second entries show us that $t_1 = 0$, the fifth entries show us that $t_2 = 0$, and the seventh entries show us that $t_3 = 0$. Thus, the only solution to the equation is $t_1 = t_2 = t_3 = 0$, and this means that our vectors are not only a spanning set, but also linearly independent. Thus, they form a basis for the solution space.

B8(g) From (f), we see that the dimension of the solution space is 3. Looking at R, we see that the rank of A is 4. And 3+4=7, the number of variables in the system $A\vec{x} = \vec{0}$.

D1 Note first that $\text{Range}(L) = \text{Col}([L])$ and $\text{Null}(L) = \text{Null}([L])$. So, $\dim(\text{Range}(L)) = \dim(\text{Col}([L])) = \text{rank}([L])$, and $\dim(\text{Null}(L)) = \dim(\text{Null}([L])) = \text{nullity}([L])$ By the rank theorem, we know that $\text{rank}([L]) + \text{nullity}([L]) = n$, and so we've shown that $\dim(\text{Range}(L)) = \dim(\text{Null}(L))$.

D3(a) By the rank theorem, we know that $\text{rank}(A) + \text{nullity}(A) = 7$, so nullity$(A) = 7 - 4 = 3$. And since dim Col$(A)$ = rank(A), the dimension of the columnspace of A is 4.

D3(b) By the rank theorem, we know that rank(A) + nullity$(A) = 4$. Since dimensions can not be negative, but they can be 0, we see that the largest possible rank or nullity is 4. So, the largest possible dimension of the nullspace of A is 4, and the largest possible rank of A is 4.

D3(c) By the rank theorem, we know that rank(A) + nullity$(A) = 5$, so rank$(A) = 5 - 3 = 2$. And since dim Row(A) = rank(A), we see that the dimension of the rowspace of A is 2.