Solution to Practice 3q

B2(a) First, we need to find $[L]$, and so we compute $L(1,0) = (1,1,0)$ and $L(0,1) = (0,2,3)$. So, $[L] = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, and now we need to find a basis for $\text{Col}([L])$ and $\text{Null}([L])$. The first step for both of these is to find the reduced row echelon form of $[L]$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{matrix} \\ (1/2)R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since both columns of the RREF have a leading 1 in them, the two columns of $[L]$ form a basis for $\text{Col}([L]) = \text{Range}(L)$, so a basis for $\text{Range}(L)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$.

Looking at the RREF, we see that the only solution to $[L]\vec{x} = \vec{0}$ is $\vec{0}$, and thus $\text{Null}([L]) = \text{Null}(L) = \{\vec{0}\}$. And the basis for $\{\vec{0}\}$ is $\emptyset$.

B2(b) First, we need to find $[M]$, and so we compute $M(1,0,0,0) = (1,0,1)$, $M(0,1,0,0) = (0,1,-2)$, $M(0,0,1,0) = (0,-2,3)$, $M(0,0,0,1) = (1,0,0)$. So $[M] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & -2 & 3 & 0 \end{bmatrix}$, and now we need to find a basis for $\text{Col}([M])$ and $\text{Null}([M])$. The first step for both of these is to find the reduced row echelon form of $[M]$:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & -2 & 3 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 3 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \\ -R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_3 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The first, second, and third columns of the RREF have a leading 1, so the first, second, and third columns of $[M]$ form a basis for $\text{Col}([M]) = \text{Range}(M)$, so a basis for $\text{Range}(M)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \right\}$.

To find a basis for the nullspace, we first note that by looking at the RREF of $[M]$, we see that the system $[M]\vec{x} = \vec{0}$ is equivalent to the system

$$\begin{array}{cccc} x_1 & & & +x_4 & = 0 \\ & x_2 & & +2x_4 & = 0 \\ & & x_3 & +x_4 & = 0 \end{array}$$

Replacing the variable x_4 with the parameter t, we get

$$\begin{array}{cccc} x_1 & & & +t & = 0 \\ & x_2 & & +2t & = 0 \\ & & x_3 & +t & = 0 \end{array}$$

And so we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

And so we have that a basis for $\text{Null}([M]) = \text{Null}(M)$ is $\left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

B5(a)(i) The number of variables in the system is the same as the number of columns in the matrix: 5

B5(a)(ii) A is already in row echelon form, so we see that the rank is 3.

B5(a)(iii) From theorem 7, we know that the dimension of the solution space is $n - r$, which in this case is 5-3=2.

B5(b)(i) The number of variables in the system is the same as the number of columns in the matrix: 4

B5(b)(ii) B is already in row echelon form, so we see that the rank is 3.

B5(b)(iii) From theorem 7, we know that the dimension of the solution space is $n - r$, which in this case is 4-3=1.

B5(c)(i) The number of variables in the system is the same as the number of columns in the matrix: 5

B5(c)(ii) C is already in row echelon form, so we see that the rank is 4.

B5(c)(iii) From theorem 7, we know that the dimension of the solution space is $n - r$, which in this case is 5-4=1.

B5(d)(i) The number of variables in the system is the same as the number of columns in the matrix: 6

B5(d)(ii) D is already in row echelon form, so we see that the rank is 3.

B5(d)(iii) From theorem 7, we know that the dimension of the solution space is $n - r$, which in this case is 6-3=3.

B6(a) First, we need to find the RREF of A:

$$\begin{bmatrix} 3 & 6 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 2R1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The non-zero rows of the RREF form a basis for Row(A), so a basis for the rowspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The columns of the RREF with a leading 1 correspond to the columns of A that form a basis for Col(A), so a basis for the columnspace is $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

To find a basis for the nullspace, we simply need to use our established technique to find the general solution to $A\vec{x} = \vec{0}$. To that end, we turn our RREF into the equivalent system:

$$\begin{matrix} x_1 & +2x_2 & & = 0 \\ & & x_3 & = 0 \end{matrix}$$

Replacing the variable x_2 with the parameter t, we get

$$\begin{matrix} x_1 & +2t & & = 0 \\ & & x_3 & = 0 \end{matrix}$$

and from this we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

And so we have found that a basis for the nullspace is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

B6(b) First, we need to find the RREF of B:

$$\begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 2 & 1 & -1 \\ 2 & 4 & 3 & -1 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 2 & 4 & 3 & -1 \end{bmatrix} R_3 - 2R - 1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The non-zero rows of the RREF form a basis for Row(B), so a basis for the rowspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

The columns of the RREF with a leading 1 correspond to the columns of B that form a basis for Col(B), so a basis for the columnspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

To find a basis for the nullspace, we simply need to use our established technique to find the general solution to $B\vec{x} = \vec{0}$. To that end, we turn our RREF into the equivalent system:

$$\begin{array}{cccccc} x_1 & & & +2x_4 & = 0 \\ & x_2 & & -2x_4 & = 0 \\ & & x_3 & +x_4 & = 0 \end{array}$$

Replacing the variable x_4 with the parameter t we get

$$\begin{array}{cccccc} x_1 & & & +2t & = 0 \\ & x_2 & & -2t & = 0 \\ & & x_3 & +t & = 0 \end{array}$$

and from this we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

And so we have found that a basis for the nullspace is $\left\{ \begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

B6(c) First, we need to find the RREF of C:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 3 & 3 \\ 1 & 1 & 3 & 6 & 8 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 2 & 2 \\ 0 & 0 & 2 & 5 & 7 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 5 & 6 \\ 0 & 0 & 2 & 5 & 7 \end{bmatrix} \begin{matrix} \\ \\ (1/2)R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & (5/2) & 3 \\ 0 & 0 & 2 & 5 & 7 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 2R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & (5/2) & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_4 \\ R_2 - 4R_4 \\ R_3 - 3R_4 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & (5/2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -3/2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & (5/2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & (5/2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The non-zero rows of the RREF form a basis for Row(C), so a basis for the rowspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The columns of the RREF with a leading 1 correspond to the columns of C that form a basis for Col(C), so a basis for the columnspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 8 \end{bmatrix} \right\}$.

To find a basis for the nullspace, we simply need to use our established technique to find the general solution to $C\vec{x} = \vec{0}$. To that end, we turn our RREF into the equivalent system:

$$\begin{array}{ccccccc} x_1 & & & +(1/2)x_4 & & = 0 \\ & x_2 & & -2x_4 & & = 0 \\ & & x_3 & +(5/2)x_4 & & = 0 \\ & & & & x_5 & = 0 \end{array}$$

Replacing the variable x_4 with the parameter t, we get

$$\begin{array}{cccccc} x_1 & & & +(1/2)t & & = 0 \\ & x_2 & & -2t & & = 0 \\ & & x_3 & +(5/2)t & & = 0 \\ & & & & x_5 & = 0 \end{array}$$

and from this we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (-1/2)t \\ 2t \\ (-5/2)t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}$$

And so we have found that a basis for the nullspace is $\left\{ \begin{bmatrix} -1/2 \\ 2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} \right\}$.