Solution to Practice 3o

B1(a) To determine if $\vec{y}_1$ is in the range of L, we need to look for $\vec{x}$ such that $[L]\vec{x} = \vec{y}_1$. That is, we need to look at solutions to

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 4 \\ 4 \end{bmatrix}$$

To solve this system, we row reduce its augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 1 & 0 & 3 & 5 \\ 2 & 1 & -1 & 4 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & -3 & -6 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{c} \\ (-1/2)R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -3 & -6 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 3R_2 \\ R_4 - 2R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & 4 & 4 \end{array}\right] \begin{array}{c} \\ \\ (-1/6)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - 4R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From our last matrix, we see that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a solution to our equation.

So, $\vec{y}_1$ is in the range of L, and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is such that

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 4 \\ 4 \end{bmatrix}$$

B1(b) To determine if $\vec{y}_2$ is in the range of L, we need to look for $\vec{x}$ such that $[L]\vec{x} = \vec{y}_2$. That is, we need to look at solutions to

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

To solve this system, we row reduce its augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & -3 \\ 2 & 1 & -1 & 2 \\ 0 & 2 & 2 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -2 & 2 & -4 \\ 0 & -3 & -3 & 0 \\ 0 & 2 & 2 & 1 \end{array}\right] \begin{array}{c} \\ (-1/2)R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -3 & -3 & 0 \\ 0 & 2 & 2 & 1 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 3R_2 \\ R_4 - 2R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 4 & 5 \end{array}\right] \begin{array}{c} \\ \\ (-1/6)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 4 & 5 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - 4R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

Since the last matrix has a bad row, we see that the equation does not have any solutions. Thus, $\vec{y}_2$ is not in the range of L.