Lecture 3q

Bases for Row(A), Col(A), and Null(A)

(pages 157-160)

Recall that the basis for a subspace S is a set of vectors that both spans S and is linearly independent. Moreover, we saw in section 2.3 that every basis for a given subspace S will have the same number of vectors, and this number is known as the dimension of the subspace.

The Basis of the Rowspace of a Matrix

The last example in Lecture 3p actually shows the technique we use to find the basis for the rowspace of a matrix–namely that we row reduce it. (I'm sure that this comes as a great surprise. Guess what we'll do to find the basis of a columnspace and nullspace...)

Theorem 3.4.5: Let B be the reduced row echelon form of an $m \times n$ matrix A. Then the non-zero rows of B form a basis for Row(A), and hence the dimension of Row(A) equals the rank of A.

Example: To find the rowspace of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, we first need to find the reduced row echelon form of A:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This last matrix is in RREF, and so its non-zero rows form a basis for the rowspace of A. Thus, $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for the rowspace of A.

Proof of Theorem 3.4.5: Theorem 3.4.4 tells us that Row(B) = Row(A), so we know that the non-zero rows of B form a spanning set for Row(A). So all we need to do is show that they are linearly independent. So, let $\vec{b}_1^T, \dots, \vec{b}_r^T$ be the non-zero rows of B, and let's look at the equation

$$t_1\vec{b}_1 + \cdots + t_r\vec{b}_r = \vec{0}$$

Now, as each of the $\vec{b}_i$ are non-zero, they have a leading term. And because B is in reduced row echelon form, this leading term is a 1. Suppose the leading 1 for $\vec{b}_i$ is in its j-th component. Then, again because B is in *reduced* row echelon form, we know that the other rows have a 0 for their j-th component. So, let's look at the j-th component of our equation:

$$t_1(\vec{b}_1)_j + \cdots + t_i(\vec{b}_i)_j + \cdots + t_r(\vec{b}_r)_j = 0$$

But since we now know that $(\vec{b}_i)_j = 1$, and $(\vec{b}_k)_j = 0$ for $k \neq i$, our equation becomes

$$0 + \cdots + t_i + \cdots + 0 = 0 \Rightarrow t_i = 0$$

So, we see that $t_i = 0$, and this holds for all $i = 1, \ldots, r$, and thus we have shown that the only solution to the equation

$$t_1\vec{b}_1 + \cdots + t_r\vec{b}_r = 0$$

is $t_1 = \cdots = t_r = 0$. And this means that the vectors $\vec{b}_1, \ldots, \vec{b}_r$ are linearly independent. And this means that we have shown that the non-zero rows of B form a basis for Row(A).

The Basis of the Columnspace of a Matrix

Yes, in order to find the basis for the columnspace of a matrix A we will look at its reduced row echelon form, but that is where the similarities with the basis for a rowspace ends. And most specifically, the basis is not simply the non-zero columns of the reduced row echelon form. Instead, we need to do something a bit more complicated to find the basis of a columnspace.

<u>Theorem 3.4.6</u>: Suppose that B is the reduced row echelon form of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis of the columnspace of A. Hence, the dimension of the columnspace equals the rank of A.

Example: To find the columnspace of $A = \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 8 \\ 3 & 1 & 2 & 9 & -3 \\ -2 & -1 & -2 & -6 & 1 \end{bmatrix}$, we first need to find its reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 8 \\ 3 & 1 & 2 & 9 & -3 \\ -2 & -1 & -2 & -6 & 1 \end{bmatrix} \begin{matrix} \\ (1/2)R_2 \\ R_3 - 3R_1 \\ R_4 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & -2 & -4 & 0 & -6 \\ 0 & 1 & 2 & 0 & 3 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 2R_2 \\ R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ (1/2)R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ R_2 - 4R_3 \\ \\ R_4 - R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in RREF, and we see that it has a leading 1 in the first, second, and fifth columns. Thus, the first, second, and fifth columns of A form a basis for the columnspace of A. That is, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for the columnspace of A.

Proof of Theorem 3.4.6: There are two facts that are going to be key to this proof. The first is that for any two row equivalent matrices C and D, we have that $C\vec{x} = \vec{0}$ if and only if $D\vec{x} = \vec{0}$, which we get from our study of homogeneous systems of linear equations. The second is Theorem 1.2.3, which says that we can remove elements of a set that can be written as a linear combination of the other elements of the set, without changing the span of the set. With these in mind, let's start the proof.

Let $A = \begin{bmatrix} \vec{a}_i & \cdots & \vec{a}_n \end{bmatrix}$, and let the reduced row echelon form of A be the matrix $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}$. Let's say that $\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}$ are the columns of B with leading 1s, and that $\vec{b}_{j_1}, \ldots, \vec{b}_{j_{n-k}}$ are the columns that do not contain leading 1s. First, let's notice that because of the structure of reduced row echelon form, the vectors $\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}$ are the first i_k standard basis vectors of R^m, so we know they are linearly independent. If we let $B^* = \begin{bmatrix} \vec{b}_{i_1} & \cdots & \vec{b}_{i_k} \end{bmatrix}$, and we let $A^* = \begin{bmatrix} \vec{a}_{i_1} & \cdots & \vec{a}_{i_k} \end{bmatrix}$ be the matrix made up of the columns of A that correspond to the columns of B that contain a leading 1, then A^* is row equivalent to B^*, and as such we know that $A^*\vec{x} = \vec{0}$ if and only if $B^*\vec{x} = \vec{0}$. As such, since the columns of B^* are linearly independent, we know that the only solution to $B^*\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, and thus the only solution to $A^*\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, which means that the columns of A^* are linearly independent.

So, we have shown that the columns of A that correspond to columns of B with leading 1s form a linearly independent set. Next, we need show that these columns, namely the columns of A^*, are a spanning set for the columnspace of A. Well, we know that the columns of A form a spanning set for the columnspace of A. As before, to do this, we will start by turning our attention to the columns of B.

Let's consider a vector $\vec{b}_{j_l}$, that is a column of B that does not correspond to a leading 1. Then its only non-zero terms occur in places where another column had a leading 1, and thus we know $\vec{b}_{j_l}$ can be written as a linear combination of the vectors $\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}$, since if $\vec{b}_{j_l} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$, then $\vec{b}_{j_l} = c_1\vec{b}_{i_1} + \cdots + c_k\vec{b}_{i_k}$. Then $c_1\vec{b}_{i_1} + \cdots + c_k\vec{b}_{i_k} - \vec{b}_{j_l} = \vec{0}$, which we can write as $\begin{bmatrix} \vec{b}_{i_1} & \cdots & \vec{b}_{i_k} & \vec{b}_{j_l} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \\ -1 \end{bmatrix} = \vec{0}$. But the matrix $\begin{bmatrix} \vec{a}_{i_1} & \cdots & \vec{a}_{i_k} & \vec{a}_{j_l} \end{bmatrix}$ is row equivalent to $\begin{bmatrix} \vec{b}_{i_1} & \cdots & \vec{b}_{i_k} & \vec{b}_{j_l} \end{bmatrix}$ so this means that we also have that $\begin{bmatrix} \vec{a}_{i_1} & \cdots & \vec{a}_{i_k} & \vec{a}_{j_l} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \\ -1 \end{bmatrix} = \vec{0}$, and thus we have $\vec{a}_{j_l} = c_1\vec{a}_{i_1} + \cdots + c_k\vec{a}_{i_k}$, so $\vec{a}_{j_l}$ can be written as a linear combination of $\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}$.

So, we've seen that the vectors $\vec{a}_{j_1}, \ldots, \vec{a}_{j_{n-k}}$ can each be written as a linear combination of the vectors in $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\}$. As such, we have that $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \text{Span}\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}, \vec{a}_{j_1}, \ldots, \vec{a}_{j_{n-k}}\} = \text{Span}\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\}$.

And so, we have shown that $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\}$ is linearly independent, and that $\text{Span}\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\} = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \text{Col}(A)$, so $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_k}\}$ is a basis for the columnspace of A.

The Basis of the Nullspace of a Matrix

In Section 2.2, we saw that if A had rank r, then the general solution of $A\vec{x} = \vec{0}$ was expressed as the spanning set of $n - r$ vectors. Shortly, we will show that these vectors are in fact linearly independent, and therefore they are a basis for the nullspace of A. But first, we introduce the following definition:

Definition: Let A be an $m \times n$ matrix. We call the dimension of the nullspace of A the **nullity** of A and denote it by $\text{nullity}(A)$.

Theorem 3.4.7: Let A be an $m \times n$ matrix with $\text{rank}(A) = r$. Then the spanning set for the general solution of the homogeneous system $A\vec{x} = \vec{0}$ obtained by the method in Chapter 2 is a basis for $\text{Null}(A)$, and the nullity of A is $n - r$.

Example: To find a basis for the nullspace of $A = \begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ -3 & 4 & -9 & -3 & 1 \\ -1 & 0 & -7 & -1 & -5 \end{bmatrix}$, we need to find the general solution to $A\vec{x} = \vec{0}$, which we start by finding the reduced row echelon form of A.

$$\begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ -3 & 4 & -9 & -3 & 1 \\ -1 & 0 & -7 & -1 & -5 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & -1 & -3 & 1 & -5 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \begin{matrix} \\ \\ (1/4)R_3 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 4 & 0 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 & 6 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The last matrix is in RREF, and thus we see that the system $A\vec{x} = \vec{0}$ is equivalent to

$$\begin{matrix} x_1 & & +7x_3 & & +6x_5 & = 0 \\ & x_2 & +3x_3 & & 4x_5 & = 0 \\ & & & x_4 & -x_5 & = 0 \end{matrix}$$

Replacing the variables x_3 and x_5 with the parameters s and t, we get

$$\begin{matrix} x_1 & & +7s & & +6t & = 0 \\ & x_2 & +3s & & 4t & = 0 \\ & & & x_4 & -t & = 0 \end{matrix}$$

and from this we see that the general solution to $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7s - 6t \\ -3s - 4t \\ s \\ t \\ t \end{bmatrix} = s \begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

And thus we have that $\left\{ \begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the nullspace of A.

Proof of Theorem 3.4.7: Let $\{\vec{v}_1, \dots, \vec{v}_{n-r}\}$ be a spanning set for the general solution of $A\vec{x} = \vec{0}$ obtained from the reduced row echelon form of A, and consider the equation

$$t_1\vec{v}_1 + \cdots + t_{n-r}\vec{v}_{n-r} = \vec{0}$$

Looking at a general term $t_i\vec{v}_i$, suppose that $\vec{v}_i$ is the vector affiliated with the parameter s_i, which came from the variable x_j. Then the j-th component of $\vec{v}_i$

is a 1. But, more importantly, the j-th component of all the other vectors will be 0. And so, if we look at the j-th equation of the system

$$t_1\vec{v}_1 + \cdots + t_i\vec{v}_i + \cdots + t_{n-r}\vec{v}_{n-r} = \vec{0}$$

we have

$$\begin{aligned} t_1(\vec{v}_1)_j + \cdots + t_i(\vec{v}_i)_j + \cdots + t_{n-r}(\vec{v}_{(n-r)})_j = 0 \quad &\Rightarrow \quad 0 + \cdots + t_i(1) + \cdots + 0 = 0 \\ &\Rightarrow \quad t_i = 0 \end{aligned}$$

But since this is true for any term $t_i\vec{v}_i$, we have shown that $t_1 = \cdots = t_{n-r} = 0$ for all $0 \leq i \leq n-r$. This means that the only solution to our vector equation is the trivial solution, and thus the set $\{\vec{v}_1, \ldots, \vec{v}_{n-r}\}$ is linearly independent. And since we already knew that it spanned the nullspace, we have shown that $\{\vec{v}_1, \ldots, \vec{v}_{n-r}\}$ is a basis for the nullspace.