Lecture 3p

Rowspace

(pages 156-157)

Well, after looking at the span of the columns of a matrix, the natural thing to look at next would be the span of the rows of a matrix:

<u>Definition</u>: Given an $m \times n$ matrix A, the **rowspace** of A is the subspace spanned by the rows of A (regarded as vectors) and is denoted Row(A).

Example: Let $A = \begin{bmatrix} 2 & 4 & 0 & -4 \\ -3 & -1 & 5 & -4 \end{bmatrix}$. Then $\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 5 \\ -4 \end{bmatrix} \right\}$.

We see that $\begin{bmatrix} 1 \\ -3 \\ -5 \\ 8 \end{bmatrix} \in \text{Row}(A)$, since $-\begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} - \begin{bmatrix} -3 \\ -1 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -5 \\ 8 \end{bmatrix}$. We see that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \notin \text{Row}(A)$ by looking for solutions to the vector equation $t_1 \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ -1 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. To do this, we will row reduce the following augmented matrix:

$$\left[\begin{array}{cc|c} 2 & -3 & 1 \\ 4 & -1 & 2 \\ 0 & 5 & 3 \\ -4 & -4 & 4 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 + 2R_1 \end{array} \sim \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 0 & 5 & 0 \\ 0 & 5 & 3 \\ 0 & -10 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 + 2R_2 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{array}\right]$$

The last two rows are both bad rows, so our vector equation does not have any solutions. This means that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is not in Row(A).

There is no counterpart in the world of linear mappings, but if we recall that the transpose of a matrix interchanges the rows and columns, we get that

$$\text{Row}(A) = \{A^T\vec{x} \in \mathbb{R}^n \mid \vec{x} \in \mathbb{R}^m\}, \text{ or } \text{Row}(A) = \text{Col}(A^T)$$

But the more interesting result for a rowspace is the following:

Theorem 3.4.4: If the $m \times n$ matrix A is row equivalent to the matrix B, then $\text{Row}(A) = \text{Row}(B)$.

Example: Let $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & -3 \\ 4 & 4 & 8 \end{bmatrix}$. Then we notice that:

$$\begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & -3 \\ 4 & 4 & 8 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ (-1/3)R_2 \\ (-1/4)R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 + R_1 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} R_2 + 2R_3$$
$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, we see that A is row equivalent to I_3. By theorem 3.4.4, this means that $\text{Row}(A) = \text{Row}(I_3) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3$.

Proof of Theorem 3.4.4: To prove theorem 3.4.4, we need to look at the effect of each type of elementary row operation on the row space.

Type 1–interchange two rows: Suppose we obtain B from A by interchanging two rows of A. Since vector addition is commutative, changing the order we list the vectors does not change the span, so $\text{Row}(B) = \text{Row}(A)$.

Type 2–multiply row i by the non-zero scalar s: Let $\vec{a}_1^T, \vec{a}_2^T, \ldots, \vec{a}_m^T$ be the rows of A, so that $\text{Row}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\}$, and let $\vec{a}_1^T, \vec{a}_2^T, \ldots, s\vec{a}_i^T, \ldots, \vec{a}_m^T$ be the rows of B. Then

$$\begin{aligned} \text{Row}(B) &= \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, s\vec{a}_i, \ldots, \vec{a}_m\} \\ &= \{t_1\vec{a}_1 + t_2\vec{a}_2 + \cdots + t_i(s\vec{a}_i) + \cdots + t_m\vec{a}_m \mid t_j \in \mathbb{R}\} \\ &= \{t_1\vec{a}_1 + t_2\vec{a}_2 + \cdots + (t_is)\vec{a}_i + \cdots + t_m\vec{a}_m \mid t_j \in \mathbb{R}\} \\ &= \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_i, \ldots, \vec{a}_m\} \\ &= \text{Row}(A) \end{aligned}$$

Type 3–add s times the i-th row to the j-th row: Let $\vec{a}_1^T, \vec{a}_2^T, \ldots, \vec{a}_m^T$ be the rows of A, so that $\text{Row}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\}$, and let $\vec{a}_1^T, \vec{a}_2^T, \ldots, \vec{a}_j^T + s\vec{a}_i^T, \ldots, \vec{a}_m^T$ be the rows of B. Then

$$
\begin{aligned}
\text{Row}(B) &= \text{Span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_j + s\vec{a}_i, \dots, \vec{a}_m\} \\
&= \{t_1\vec{a}_1 + t_2\vec{a}_2 + \cdots + t_i\vec{a}_i + \cdots + t_j(\vec{a}_j + s\vec{a}_i) + \cdots + t_m\vec{a}_m \mid t_k \in \mathbb{R}\} \\
&= \{t_1\vec{a}_1 + t_2\vec{a}_2 + \cdots + (t_i + st_j)\vec{a}_i + \cdots + (t_j s)\vec{a}_j + \cdots + t_m\vec{a}_m \mid t_k \in \mathbb{R}\} \\
&= \text{Span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_i, \dots, \vec{a}_j, \dots, \vec{a}_m\} \\
&= \text{Row}(A)
\end{aligned}
$$