Lecture 3o

Columnspace

(pages 153-156)

Next up in our list of new and interesting subspaces of $\mathbb{R}^n$ is the columnspace of a matrix.

Definition: Let A be an $m \times n$ matrix, and let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n \in \mathbb{R}^m$ be the columns of A. Then the **columnspace** of A, written Col(A), is Span$\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$.

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 6 \\ 1 & 2 & 3 \\ -1 & 5 & 9 \end{bmatrix}$. Then $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \\ 9 \end{bmatrix} \right\}$.

We see that $\begin{bmatrix} 8 \\ 7 \\ 2 \\ 11 \end{bmatrix} \in \mathrm{Col}(A)$, since $5\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 2 \\ 11 \end{bmatrix}$.

To see that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in Col(A), we need to look for solutions to the vector equation

$$t_1\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ 3 \\ 2 \\ 5 \end{bmatrix} + t_3\begin{bmatrix} 3 \\ 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

To solve this system, we need to row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 2 & 3 & 6 & 1 \\ 1 & 2 & 3 & 1 \\ -1 & 5 & 9 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 3 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 5 & 12 & 2 \end{array}\right] R_2 \updownarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 5 & 12 & 2 \end{array}\right] \begin{array}{c} \\ \\ R_3 - (3/2)R_2 \\ R_4 - (5/2)R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 12 & 2 \end{array}\right]$$

The last matrix is not in row echelon form, but we already see that row 3 is a bad row, so the system is inconsistent. Thus, our vector equation does not have any solutions, and as such, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in Col(A).

The definition of a columnspace is pretty straightforward: the columnspace is a subspace formed from the columns. However, this is not the definition given in the textbook. The textbook definition is of course the same as mine, so I will now give the textbook definition and show that they are the same.

Definition: The **columnspace** of an $m \times n$ matrix A is the set $\mathrm{Col}(A)$ defined by

$$\mathrm{Col}(A) = \{A\vec{x} \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\}$$

To see that these really are the same set, you need to notice that $A\vec{x}$ is simply a linear combination of the columns of A. And the best way to do that is to use block multiplication. Since $A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix}$, we have that $A\vec{x} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n]$.

Now, if my definition is so straightforward, why does the textbook use something else? Well, the textbook definition lets us view the columnspace as something related to linear mappings, instead of simply a property of a matrix. And, as linear mapping, what we are looking at is the following:

Definition: The **range** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined to be the set

$$\mathrm{Range}(L) = \{L(\vec{x}) \in \mathbb{R}^m \mid \vec{x} \in \mathbb{R}^n\}$$

The range differs from the codomain in that the codomain is simply the vector space that elements of the domain get mapped to, while the range is a list of the vectors that actually get mapped to. So, there may be elements of the codomain that are not in the range. Consider the following examples.

Example: Determine whether or not $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$ are in the range of the linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $L(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$.

$\vec{v}$: To determine if $\vec{v}$ is in the range of L, we need to try to find a solution to the vector equation $\begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We could set this up as a system of linear equations, but this particular equation is easy to solve with some quick observations. First of all, we notice that the first row tells us that $x_1 = 1$. Using this in the second row, we see that $x_1 + x_2 = 1$, so $x_2 = 1 - x_1 = 1 - 1 = 0$.

Plugging these values of x_1 and x_2 into the third row yields the true equation $1 - 0 = 1$. So, we see that $L(1, 0) = (1, 1, 1) = \vec{v}$, so $\vec{v}$ IS in the range of L.

$\vec{y}$: To determine if $\vec{y}$ is in the range of L, we need to try to find a solution to the vector equation $\begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$. The first row tells us that $x_1 = 2$. Using this in the second row, we see that $x_1 + x_2 = 3$, so $x_2 = 3 - x_1 = 3 - 2 = 1$. Plugging these values of x_1 and x_2 into the third row yields the false equation $2 - 1 = -2$. So, we see that there are no vectors $\vec{x}$ such that $L(x_1, x_2) = (2, 3, -2)$, so $\vec{y}$ is NOT in the range of L.

Example: Show that the range of $L(x_1, x_2, x_3) = (x_1 + 2x_2, x_1 + 3x_3)$ is $\mathbb{R}^2$.

That is, for any $\vec{y} \in \mathbb{R}^2$, we need to find an $\vec{x} \in \mathbb{R}^3$ such that $L(\vec{x}) = \vec{y}$. As a vector equations, this is

$$\begin{bmatrix} x_1 + 2x_2 \\ x_1 + 3x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

This corresponds to the following system of linear equations:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & & & = & y_1 \\ x_1 & & & + & 3x_3 & = & y_2 \end{array}$$

This system has augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & y_1 \\ 1 & 0 & 3 & y_2 \end{array}\right]$. If we subtract the first row from the second row, then we get the matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & y_1 \\ 0 & -2 & 3 & y_2 - y_1 \end{array}\right]$, which is in row echelon form. And so we see that the rank of the coefficient matrix is the same as the rank of the augmented matrix, and thus the system is consistent. That is, the system does have a solution, no matter what y_1, y_2 we pick, so every element $\vec{y} \in \mathbb{R}^2$ is in the range of L.

These examples help us notice the following:

Theorem 3.4.3: The system of equations $A\vec{x} = \vec{b}$ is consistent if and and only if $\vec{b}$ is in the range of the linear mapping L with standard matrix A, or equivalently, if and only if $\vec{b}$ is in the columnspace of A.

The proof of Theorem 3.4.3 is straightforward, and in the textbook, so I won't repeat it here. As one last comment, note that if we think of a columnspace from the perspective of being the span of the columns of a matrix, it doesn't really make sense to talk about the columnspace of a linear mapping L. But we do of course have that $\text{Range}(L) = \text{Col}([L])$, and so we see that $\text{Range}(L)$ is a subspace of $\mathbb{R}^m$.

Example: Let L be the linear mapping defined by $L(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 + x_3, x_3 + x_4)$. Find Range($L$).

First, we must find $[L]$, and to do this we note that

$$L(1,0,0,0) = (1,0,0) \qquad L(0,1,0,0) = (1,1,0)$$
$$L(0,0,1,0) = (0,1,1) \qquad L(0,0,0,1) = (0,0,1)$$

And so $[L] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Then $\text{Range}(L) = \text{Col}([L]) =$ $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.