Lecture 3n

Solution Sets

(pages 152-153)

In the last lecture, we revisited the topic of homogeneous systems of linear equations, noting that the general solution to such a system is a subspace of $\mathbb{R}^n$. But what about an arbitrary system of linear equations? Well, if S is the solution set to the system of linear equations given by $A\vec{x} = \vec{b}$, where $\vec{b} \neq \vec{0}$, then S is NOT A SUBSPACE of $\mathbb{R}^n$. Not "maybe" or "we don't know". Definitely "not a subspace." The easiest way to see this is to note that $\vec{0} \notin S$, since $A\vec{0} = \vec{0}$, so $A\vec{0} \neq \vec{b}$. While this technically does not contradict the definition of being a subspace, we discussed back in Lecture 1g why the subspace definition requires that $\vec{0}$ be an element of every subspace.

Okay, lecture done, you must be thinking at this point. Well, you're almost right. But before we return to looking at subspaces, we can take advantage of our study of linear mappings to formalize a result that you might have noticed in our earlier work solving systems of equations.

<u>Theorem 3.4.2:</u> Let $\vec{p}$ be a solution of the system of linear equations $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$.

(1) If $\vec{v}$ is any other solution of the same system, then $A(\vec{p} - \vec{v}) = \vec{0}$, so that $\vec{p} - \vec{v}$ is a solution of the corresponding homogeneous solution $A\vec{x} = \vec{0}$.

(2) If $\vec{h}$ is any solution of the corresponding system $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of the system $A\vec{x} = \vec{b}$.

The proof of Theorem 3.4.2 follows easily from the linearity properties of A, and is given in the textbook, so I won't reproduce it here. I would, however, like to add another result, that seems to be missing from the textbook:

<u>Addition to Theorem 3.4.2:</u> Let $\vec{p}$ be a solution of the system of linear equations $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$, and let Null(A) be the solution set for the corresponding homogeneous system $A\vec{x} = \vec{0}$. Then the solution set for $A\vec{x} = \vec{b}$ is

$$S = \{\vec{x} + \vec{p} \mid \vec{x} \in \text{Null}(A)\}$$

<u>Proof of Addition to Theorem 3.4.2</u> To prove this fact, we need to show two things: (1) every element of S (that is, every vector of the form $\vec{x} + \vec{p}$ where $\vec{x} \in \text{Null}(A)$) is a solution of $A\vec{x} = \vec{b}$, and (2) every solution to $A\vec{x} = \vec{b}$ is in S, that is of the form $\vec{y} + \vec{p}$ where $\vec{y} \in \text{Null}(A)$. Well, (1) follows directly from part (2) of Theorem 3.4.2. For (2), suppose $\vec{h}$ is a solution to $A\vec{x} = \vec{b}$. Then Theorem 1.5.2,(1) says that $\vec{h} - \vec{p} \in \text{Null}(A)$. So, let $\vec{h} - \vec{p}$ be our "$\vec{y}$", and we have that $\vec{h} = (\vec{h} - \vec{p}) + \vec{p}$, where $(\vec{h} - \vec{p}) \in \text{Null}(A)$.

So, my addition to Theorem 3.4.2 follows easily from Theorem 3.4.2, but I like this result because it notes that the solution set of a non-homogeneous system is simply a translation of the solution set of the homogeneous system. So, if the solution to the homogeneous system was a line through the origin, then the solution to the non-homogeneous solution is a line through $\vec{p}$. If the solution to the homogeneous system was a hyperplane through the origin, then the solution to the non-homogeneous solution is a hyperplane through $\vec{p}$. It also means that if you know that the general solution to $A\vec{x} = \vec{0}$ is $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then the general solution to $A\vec{x} = \vec{b}$ would be $t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_n\vec{v}_n + \vec{p}$. Of course, we don't often find $\vec{p}$ without also finding $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, but I still think its a nice idea.