Lecture 3m

Nullspace

(pages 150-152)

Back in Section 1.2 (and, more specifically, in Lecture 1g) we defined a subspace of $\mathbb{R}^n$ to be a non-empty subset of $\mathbb{R}^n$ that is closed under addition and scalar multiplication. Moreover, we found that any easy way to define a subspace was as the span of a set of vectors. Now that we know about matrices and linear mappings, we can look at more collections of subspaces from $\mathbb{R}^n$. We will start by looking at the nullspace of a linear mapping:

Definition: The **nullspace** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all vectors in $\mathbb{R}^n$ whose image under L is the zero vector $\vec{0}$. We write

$$\text{Null}(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{0}\}$$

The **nullspace** of an $m \times n$ matrix A is

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

So when we talk about the nullspace of a matrix, we are thinking of the matrix as a linear mapping. And we easily see that $\text{Null}(L) = \text{Null}([L])$.

Remark The word **kernel** and the notation $\ker(L)$ or $\ker(A)$ is often used instead of the term nullspace. The textbook also first defines the nullspace of A as the **solution space** of A.

Example: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is in the nullspace of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in the nullspace of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example: Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$L(x_1, x_2) = (x_1 - x_2, 2x_1 - 2x_2, 3x_1 - 3x_2)$$

Then $(-2, 2)$ is in the nullspace of L, since

$$L(-2, 2) = (-2 - (-2), -4 - (-4), -6 - (-6)) = (0, 0, 0)$$

but $(0, 1)$ is not in the nullspace of L, since

$$L(0, 1) = (-1, -2, -3) \neq (0, 0, 0)$$

Modified Theorem 3.4.1: Let A be an $m \times n$ matrix. Then $\text{Null}(A)$ is a subspace of $\mathbb{R}^n$.

Proof of Theorem 3.4.1: First, we note that $\vec{0} \in \text{Null}(A)$, since $A\vec{0} = \vec{0}$ for any matrix A. So $\text{Null}(A)$ is non-empty. Now, suppose $\vec{x}, \vec{y} \in \text{Null}(A)$, and let $t \in \mathbb{R}$. Then $A\vec{x} = \vec{0} = A\vec{y}$. Using the linearity properties of matrix multiplication, we have that $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$ and $A(t\vec{x}) = tA\vec{x} = t\vec{0} = \vec{0}$. So $\vec{x} + \vec{y} \in \text{Null}(A)$ and $t\vec{x} \in \text{Null}(A)$. Thus, we see that $\text{Null}(A)$ is a subspace of $\mathbb{R}^n$.

Theorem 3.4.1 justifies the use of "space" in the word nullspace. Since $\text{Null}(L) = \text{Null}([L])$, we also have that $\text{Null}(L)$ is a subspace of $\mathbb{R}^n$.

So, we have defined the nullspace, but how do we find the nullspace? This is best seen through examples:

Example: Find the nullspace of $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 4 \\ 3 & 9 & 6 \end{bmatrix}$.

This is the same as finding the general solution to the homogeneous system $A\vec{x} = \vec{0}$, which has coefficient matrix A. So, the first step is to row reduce A.

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 4 \\ 3 & 9 & 6 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in RREF, and is equivalent to the system

$$\begin{array}{rcrcl} x_1 & & - & 7x_3 & = & 0 \\ & x_2 & + & 3x_3 & = & 0 \end{array}$$

Replacing the variable x_3 with the parameter t, we get

$$\begin{array}{ccccccc} x_1 & & & - & 7t & = & 0 \\ & & x_2 & + & 3t & = & 0 \end{array}$$

So we see that the general solution to the homogeneous system $A\vec{x} = \vec{0}$ is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$. That is, we have $\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \right\}$.

Example: Find the nullspace of L, where L is defined by $L(x_1, x_2, x_3) = (x_1 + 2x_2, x_1 - 2x_2 + 4x_3)$.

We can start this questions in two different ways. The first is to take the question at face value, and look for all solutions to $\begin{bmatrix} x_1 + 2x_2 \\ x_1 - 2x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is equivalent to the following system of linear equations:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & & & & 0 \\ x_1 & - & 2x_2 & + & 4x_3 & = & 0 \end{array}$$

To solve this system, we need to row reduce the coefficient matrix $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 4 \end{bmatrix}$. But wait–this coefficient matrix is actually $[L]$. So, the other way we could have started this question would be use the fact that $\text{Null}(L) = \text{Null}([L])$. To find $[L]$, we note that $L(1,0,0) = (1,1)$, $L(0,1,0) = (2,-2)$, and $L(0,0,1) = (0,4)$. Then $[L] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 4 \end{bmatrix}$, as we noted earlier. And then we would want to find the general solution to $[L]\vec{x} = \vec{0}$, which is equivalent to finding the general solution to the system of linear equations given earlier. So, no matter which way we start the question, we end up looking for the general solution to $[L]\vec{x} = \vec{0}$. The next step is to row reduce $[L]$:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 4 \end{bmatrix} \begin{matrix} \\ (-1/4)R_2 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

This last matrix is in RREF, and is equivalent to the system

$$\begin{array}{ccccccc} x_1 & & & + & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 0 \end{array}$$

Replacing the variable x_3 with the parameter t, we get

$$\begin{array}{ccccccc} x_1 & & & + & 2t & = & 0 \\ & & x_2 & - & t & = & 0 \end{array}$$

So, we see that the general solution to $[L]\vec{x} = \vec{0}$ is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. And so we see that $\text{Null}(L) = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$.