Solution to Practice 3l

B1(a) $R_{\frac{-\pi}{2}} = \begin{bmatrix} \cos\frac{-\pi}{2} & -\sin\frac{-\pi}{2} \\ \sin\frac{-\pi}{2} & \cos\frac{-\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

B1(b) $R_{(-\pi)} = \begin{bmatrix} \cos(-\pi) & -\sin(-\pi) \\ \sin(-\pi) & \cos(-\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

B1(c) $R_{\frac{3\pi}{4}} = \begin{bmatrix} \cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4} \\ \sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$

B1(d) $R_{\frac{-\pi}{5}} = \begin{bmatrix} \cos\frac{-\pi}{5} & -\sin\frac{-\pi}{5} \\ \sin\frac{-\pi}{5} & \cos\frac{-\pi}{5} \end{bmatrix} \approx \begin{bmatrix} .809 & .588 \\ -.588 & .809 \end{bmatrix}$

B2(a) $S = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix}$

B2(b) $R_\theta S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix} = \begin{bmatrix} \cos\theta & -0.6\sin\theta \\ \sin\theta & 0.6\cos\theta \end{bmatrix}$

B2(c) $SR_{\frac{\pi}{4}} = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} =$
$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0.6\left(\frac{\sqrt{2}}{2}\right) & 0.6\left(\frac{\sqrt{2}}{2}\right) \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 0.6 & 0.6 \end{bmatrix}$

B3(a) Reading off from the equation $x_1 - 5x_2 = 0$, we see that $\vec{n} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$. We will need to compute $\text{proj}_{\vec{n}}\vec{e}_1$ and $\text{proj}_{\vec{n}}\vec{e}_2$, and to that end we first note the following:

$$||\vec{n}||^2 = 1^2 + (-5)^2 = 26 \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -5 \end{bmatrix} = 1 \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -5$$

Then $\text{proj}_{\vec{n}}\vec{e}_1 = \frac{\vec{e}_1 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{1}{26}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 1/26 \\ -5/26 \end{bmatrix}$,
and $\text{proj}_{\vec{n}}\vec{e}_2 = \frac{\vec{e}_2 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{-5}{26}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -5/26 \\ 25/26 \end{bmatrix}$

And now we can compute that $\text{refl}_{\vec{n}}\vec{e}_1 = \vec{e}_1 - 2\text{proj}_{\vec{n}}\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1/26 \\ -5/26 \end{bmatrix} = \begin{bmatrix} 24/26 \\ 10/26 \end{bmatrix}$,
and $\text{refl}_{\vec{n}}\vec{e}_2 = \vec{e}_2 - 2\text{proj}_{\vec{n}}\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -5/26 \\ 25/26 \end{bmatrix} = \begin{bmatrix} 10/26 \\ -24/26 \end{bmatrix}$. This means that

$$[\text{refl}_{\vec{n}}] = \begin{bmatrix} 24/26 & 10/26 \\ 10/26 & -24/26 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 12 & 5 \\ 5 & -12 \end{bmatrix}$$

B3(b) Reading off from the equation $3x_1 + 4x_2 = 0$, we see that $\vec{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. We will need to compute $\text{proj}_{\vec{n}}\vec{e}_1$ and $\text{proj}_{\vec{n}}\vec{e}_2$, and to that end we first note the following:

$$||\vec{n}||^2 = 3^2 + 4^2 = 25 \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 4$$

Then $\text{proj}_{\vec{n}}\vec{e}_1 = \frac{\vec{e}_1 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{3}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9/25 \\ 12/25 \end{bmatrix}$,
and $\text{proj}_{\vec{n}}\vec{e}_2 = \frac{\vec{e}_2 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{4}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 12/25 \\ 16/25 \end{bmatrix}$

And now we can compute that $\text{refl}_{\vec{n}}\vec{e}_1 = \vec{e}_1 - 2\text{proj}_{\vec{n}}\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 9/25 \\ 12/25 \end{bmatrix} = \begin{bmatrix} 7/25 \\ -24/25 \end{bmatrix}$,
and $\text{refl}_{\vec{n}}\vec{e}_2 = \vec{e}_2 - 2\text{proj}_{\vec{n}}\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 12/25 \\ 16/25 \end{bmatrix} = \begin{bmatrix} -24/25 \\ -7/25 \end{bmatrix}$. This means that

$$[\text{refl}_{\vec{n}}] = \begin{bmatrix} 7/25 & -24/25 \\ -24/25 & -7/25 \end{bmatrix}$$

B4(a) Reading off from the equation $x_1 - 3x_2 - x_3 = 0$, we see that $\vec{n} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$. We will need to compute $\text{proj}_{\vec{n}}\vec{e}_1$, $\text{proj}_{\vec{n}}\vec{e}_2$, and $\text{proj}_{\vec{n}}\vec{e}_3$, and to that end we first note the following:

$$||\vec{n}||^2 = 1^2 + (-3)^2 + (-1)^2 = 11$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = 1 \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = -3 \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = -1$$

Then $\text{proj}_{\vec{n}}\vec{e}_1 = \frac{\vec{e}_1 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{1}{11}\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/11 \\ -3/11 \\ -1/11 \end{bmatrix}$,

$\text{proj}_{\vec{n}}\vec{e}_2 = \frac{\vec{e}_2\cdot\vec{n}}{||\vec{n}||^2}\vec{n} = \frac{-3}{11}\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/11 \\ 9/11 \\ 3/11 \end{bmatrix}$,
and $\text{proj}_{\vec{n}}\vec{e}_3 = \frac{\vec{e}_3\cdot\vec{n}}{||\vec{n}||^2}\vec{n} = \frac{-1}{11}\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/11 \\ 3/11 \\ 1/11 \end{bmatrix}$.

And now we can compute that $\text{refl}_{\vec{n}}\vec{e}_1 = \vec{e}_1 - 2\text{proj}_{\vec{n}}\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1/11 \\ -3/11 \\ -1/11 \end{bmatrix} = \begin{bmatrix} 9/11 \\ 6/11 \\ 2/11 \end{bmatrix}$,
$\text{refl}_{\vec{n}}\vec{e}_2 = \vec{e}_2 - 2\text{proj}_{\vec{n}}\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -3/11 \\ 9/11 \\ 3/11 \end{bmatrix} = \begin{bmatrix} 6/11 \\ -7/11 \\ -6/11 \end{bmatrix}$,
and $\text{refl}_{\vec{n}}\vec{e}_3 = \vec{e}_3 - 2\text{proj}_{\vec{n}}\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1/11 \\ 3/11 \\ 1/11 \end{bmatrix} = \begin{bmatrix} 2/11 \\ -6/11 \\ 9/11 \end{bmatrix}$. This means that

$$[\text{refl}_{\vec{n}}] = \begin{bmatrix} 9/11 & 6/11 & 2/11 \\ 6/11 & -7/11 & -6/11 \\ 2/11 & -6/11 & 9/11 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 9 & 6 & 2 \\ 6 & -7 & -6 \\ 2 & -6 & 0 \end{bmatrix}$$

B4(b) Reading off from the equation $2x_1 + x_2 - x_3 = 0$, we see that $\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.
We will need to compute $\text{proj}_{\vec{n}}\vec{e}_1$, $\text{proj}_{\vec{n}}\vec{e}_2$, and $\text{proj}_{\vec{n}}\vec{e}_3$, and to that end we first note the following:

$$||\vec{n}||^2 = 2^2 + 1^2 + (-1)^2 = 6$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2 \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 1 \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = -1$$

Then $\text{proj}_{\vec{n}}\vec{e}_1 = \frac{\vec{e}_1\cdot\vec{n}}{||\vec{n}||^2}\vec{n} = \frac{2}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/6 \\ 2/6 \\ -2/6 \end{bmatrix}$,
$\text{proj}_{\vec{n}}\vec{e}_2 = \frac{\vec{e}_2\cdot\vec{n}}{||\vec{n}||^2}\vec{n} = \frac{1}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/6 \\ 1/6 \\ -1/6 \end{bmatrix}$,

and $\text{proj}_{\vec{n}}\vec{e}_3 = \frac{\vec{e}_3 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{-1}{6}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/6 \\ -1/6 \\ 1/6 \end{bmatrix}$.

And now we can compute that $\text{refl}_{\vec{n}}\vec{e}_1 = \vec{e}_1 - 2\text{proj}_{\vec{n}}\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 4/6 \\ 2/6 \\ -2/6 \end{bmatrix} = \begin{bmatrix} -2/6 \\ -4/6 \\ 4/6 \end{bmatrix}$,

$\text{refl}_{\vec{n}}\vec{e}_2 = \vec{e}_2 - 2\text{proj}_{\vec{n}}\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 2/6 \\ 1/6 \\ -1/6 \end{bmatrix} = \begin{bmatrix} -4/6 \\ 4/6 \\ 2/6 \end{bmatrix}$,

and $\text{refl}_{\vec{n}}\vec{e}_3 = \vec{e}_3 - 2\text{proj}_{\vec{n}}\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -2/6 \\ -1/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 4/6 \\ 2/6 \\ 4/6 \end{bmatrix}$. This means that

$$[\text{refl}_{\vec{n}}] = \begin{bmatrix} -2/6 & -4/6 & 4/6 \\ -4/6 & 4/6 & 2/6 \\ 4/6 & 2/6 & 4/6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

B5(a) First we note that $[C] = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. To find [inj], we first compute

$$\text{inj}(1,0,0) = (0,1,0,0,0) \quad \text{inj}(0,1,0) = (0,0,0,1,0) \quad \text{inj}(0,0,1) = (0,0,0,0,1)$$

And so we see that $[\text{inj}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

And thus the matrix for $\text{inj} \circ C = [\text{inj}][C] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

B5(b) The first thing we need to do is find $[S]$, and to do that we need to do the following calculations:

$$S(1,0,0) = (1,0,0) \quad S(0,1,0) = (0,1,0) \quad S(0,0,1) = (0,-2,1)$$

Then we have $[S] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, and thus

$$[C \circ S] = [C][S] = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & -2/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$[S \circ C] = [S][C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & -2/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

B5(c) The The first thing we need to do is find $[T]$, and to do that we need to do the following calculations:

$$T(1,0,0) = (1,0,0) \quad T(0,1,0) = (3,1,0) \quad T(0,0,1) = (0,0,1)$$

Then we have $[T] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and thus

$$[S \circ T] = [S][T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T \circ S] = [T][S] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$