Solution to Practice 3j

B4(a) Domain: $\mathbb{R}^3$. Codomain: $\mathbb{R}^3$. To find the standard matrix, we find the image of the standard basis vectors:

$$L(1,0,0) = (2,0,4) \qquad L(0,1,0) = (-3,1,-5) \qquad L(0,0,1) = (0,0,0)$$

So $[L] = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 0 \\ 4 & -5 & 0 \end{bmatrix}$

B4(b) Domain: $\mathbb{R}^4$. Codomain: $\mathbb{R}^3$. To find the standard matrix, we find the image of the standard basis vectors:

$$K(1,0,0,0) = (2,-1,0) \quad K(0,1,0,0) = (0,-2,3)$$
$$K(0,0,1,0) = (-1,2,1) \quad K(0,0,0,1) = (3,1,0)$$

So $[K] = \begin{bmatrix} 2 & 0 & -1 & 3 \\ -1 & -2 & 2 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}$

B4(c) Domain: $\mathbb{R}^3$. Codomain: $\mathbb{R}^3$. To find the standard matrix, we find the image of the standard basis vectors:

$$M(1,0,0) = (-1,0,5) \qquad M(0,1,0) = (0,0,1) \qquad M(0,0,1) = (1,0,0)$$

So $[M] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix}$

B8 Since $\text{proj}_{\vec{v}}$ is linear, we can apply Theorem 3 to find $[\text{proj}_{\vec{v}}]$. This means that we need to look at the image of the standard basis vectors. But first, let's do some calculations that we'll need.

$||\vec{v}||^2 = 1^2 + (-3)^2 = 1 + 9 = 10$, $\vec{e}_1 \cdot \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = (1)(1) + (0)(-3) = 1$,
and $\vec{e}_2 \cdot \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = (0)(1) + (1)(-3) = -3$.

And now we see that $\text{proj}_{\vec{v}}\vec{e}_1 = (1/10)\vec{v} = (1/10, -3/10)$ and $\text{proj}_{\vec{v}}\vec{e}_2 = (-3/10)\vec{v} = (-3/10, 9/10)$.

Thus, $[\text{proj}_{\vec{v}}] = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$

B11 Since $\text{perp}_{\vec{v}}$ is linear, we can apply Theorem 3 to find $[\text{perp}_{\vec{v}}]$. This means that we need to look at the image of the standard basis vectors. But first, let's do some calculations that we'll need.

$||\vec{v}||^2 = 1^2 + (-2)^2 + 2^2 = 9$, $\vec{e}_1 \cdot \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 1$,
$\vec{e}_2 \cdot \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = -2$, and $\vec{e}_3 \cdot \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 2$.

And now we see that

$$\text{perp}_{\vec{v}}\vec{e}_1 = (1,0,0) - (1/9)\vec{v} = (1,0,0) - (1/9, -2/9, 2/9) = (8/9, 2/9, -2/9),$$
$$\text{perp}_{\vec{v}}\vec{e}_2 = (0,1,0) - (-2/9)\vec{v} = (0,1,0) + (2/9, -4/9, 4/9) = (2/9, 5/9, 4/9),$$

and

$$\text{perp}_{\vec{v}}\vec{e}_3 = (0,0,1) - (2/9)\vec{v} = (0,0,1) - (2/9, -4/9, 4/9) = (-2/9, 4/9, 5/9).$$

So we have $[\text{perp}_{\vec{v}}] = \begin{bmatrix} 8/9 & 2/9 & -2/9 \\ 2/9 & 5/9 & 4/9 \\ -2/9 & 4/9 & 5/9 \end{bmatrix}$

D3 We show that $\text{DOT}_{\vec{v}}$ preserves addition as follows:

$$\begin{aligned} \text{DOT}_{\vec{v}}(\vec{x} + \vec{y}) &= \vec{v} \cdot (\vec{x} + \vec{y}) \\ &= \vec{v} \cdot \vec{x} + \vec{v} \cdot \vec{y} && \text{(Theorem 1,(3), p.31)} \\ &= \text{DOT}_{\vec{v}}\vec{x} + \text{DOT}_{\vec{v}}\vec{y} \end{aligned}$$

We show that $\text{DOT}_{\vec{v}}$ preserves scalar multiplication as follows:

$$\begin{aligned} \text{DOT}_{\vec{v}}(t\vec{x}) &= \vec{v} \cdot (t\vec{x}) \\ &= t(\vec{v} \cdot \vec{x}) && \text{(Theorem 1,(4), p.31)} \\ &= t\text{DOT}_{\vec{v}}\vec{x} \end{aligned}$$

And since $\text{DOT}_{\vec{v}}$ preserves addition and scalar multiplication, we see that $\text{DOT}_{\vec{v}}$ is a linear mapping. Its codomain is $\mathbb{R}$. Moreover, the i-th entry in $[\text{DOT}_{\vec{v}}]$ is $\text{DOT}_{\vec{v}}\vec{e}_i = v_i$. So $[\text{DOT}_{\vec{v}}] = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \vec{v}^T$.