Solution to Practice 3i

B3(a) The domain of f is $\mathbb{R}^2$, and the codomain of f is $\mathbb{R}^2$ as well. We show that f preserves addition as follows:

$$\begin{aligned} f(x_1+y_1, x_2+y_2) &= (2(x_2+y_2), (x_1+y_1)-(x_2-y_2)) \\ &= (2x_2+2y_2, (x_1-x_2)+(y_1-y_2)) \\ &= (2x_2, x_1-x_2)+(2y_2, y_1-y_2) \\ &= f(x_1,x_2)+f(y_1,y_2) \end{aligned}$$

We show that f preserves scalar multiplication as follows:

$$\begin{aligned} f(tx_1, tx_2) &= (2tx_2, tx_1-tx_2) \\ &= (t(2x_2), t(x_1-x_2)) \\ &= t(2x_2, x_1-x_2) \\ &= tf(x_1,x_2) \end{aligned}$$

Thus, f IS a linear mapping.

B3(b) The domain of g is $\mathbb{R}^2$ and the codomain of g is $\mathbb{R}^2$ as well. But g is NOT a linear mapping. As a counterexample, consider that $g(0,0)=(1,0)$ and $g(1,\pi)=(-1,\pi^3)$, so $g(0,0)+g(1,\pi)=(1,0)+(-1,\pi^3)=(0,\pi^3)$. But $g((0,0)+(1,\pi))=g(1,\pi)=(-1,\pi^3)\neq g(0,0)+g(1,\pi)$.

B3(c) The domain of h is $\mathbb{R}^3$ and the codomain of h is $\mathbb{R}^3$ as well. We show that h preserves addition as follows:

$$\begin{aligned} h(x_1+y_1, x_2+y_2, x_3+y_3) &= (0,0,(x_1+y_1)+(x_2+y_2)+(x_3+y_3)) \\ &= (0,0,(x_1+x_2+x_3)+(y_1+y_2+y_3)) \\ &= (0,0,x_1+x_2+x_3)+(0,0,y_1+y_2+y_3) \\ &= h(x_1,x_2,x_3)+h(y_1,y_2,y_3) \end{aligned}$$

We show that h preserves scalar multiplication as follows:

$$\begin{aligned} h(tx_1, tx_2, tx_3) &= (0,0,tx_1+tx_2+tx_3) \\ &= (t(0), t(0), t(x_1+x_2+x_3)) \\ &= t(0,0,x_1+x_2+x_3) \\ &= th(x_1,x_2,x_3) \end{aligned}$$

Thus, h IS a linear mapping.

B3(d) The domain of k is $\mathbb{R}^3$, and the codomain of k is $\mathbb{R}^3$ as well. We see that k preserves addition as follows:

$$\begin{aligned} k(x_1+y_1, x_2+y_2, x_3+y_3) &= (0,0,0) \\ &= (0,0,0)+(0,0,0) \\ &= k(x_1,x_2,x_3)+k(y_1,y_2,y_3) \end{aligned}$$

We show that h preserves scalar multiplication as follows:

$$\begin{aligned} k(tx_1, tx_2, tx_3) &= (0,0,0) \\ &= (t(0), t(0), t(0)) \\ &= t(0,0,0) \\ &= tk(x_1, x_2, x_3) \end{aligned}$$

Thus, k IS a linear mapping.

B3(e) The domain of l is $\mathbb{R}^4$, and the codomain of l is $\mathbb{R}^3$. But l is NOT a linear mapping. As a counterexample, consider that $l(1,0,0,0) = (1,1,0)$ and $l(0,1,0,0) = (0,1,0)$, so $l(1,0,0,0) + l(0,1,0,0) = (1,1,0) + (0,1,0) = (1,2,0)$. But $l((1,0,0,0) + (0,1,0,0)) = l(1,1,0,0) = (1,1,0) \neq l(1,0,0,0) + l(0,1,0,0)$.

B3(f) The domain of m is $\mathbb{R}^4$, and the codomain of m is $\mathbb{R}^1 = \mathbb{R}$. We show that m preserves addition as follows:

$$\begin{aligned} m(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4) &= ((x_1+y_1) + (x_2+y_2) - (x_3+y_3)) \\ &= ((x_1+x_2-x_3) + (y_1+y_2-y_3)) \\ &= (x_1+x_2-x_3) + (y_1+y_2-y_3) \\ &= m(x_1,x_2,x_3,x_4) + m(y_1,y_2,y_3,y_4) \end{aligned}$$

We show that m preserves scalar multiplication as follows:

$$\begin{aligned} m(tx_1, tx_2, tx_3, tx_4) &= (tx_1 + tx_2 - tx_3) \\ &= t(x_1 + x_2 - x_3) \\ &= tm(x_1, x_2, x_3, x_4) \end{aligned}$$

Course Author question: Let $f(\vec{0}) = \vec{x}$, where $\vec{x} \neq \vec{0}$. Then $2\vec{x} \neq \vec{x}$. And we have $f(2(\vec{0})) = f(\vec{0}) = \vec{x}$, but $2f(\vec{0}) = 2\vec{x}$. So $f(2(\vec{0})) \neq 2f(\vec{0})$, and so we see that f does not preserve scalar multiplication.