Lecture 3l

Geometrical Transformations

(pages 143-148)

In the previous section of the text, we defined linear mappings, and noted that some mappings we had previously looked at ($\text{proj}_{\vec{n}}$, $\text{perp}_{\vec{n}}$, $\text{DOT}_{\vec{n}}$) were all examples of linear mappings. The purpose of section 3.3 is to introduce you to some other commonly used linear mappings, all of which have a visual interpretation as some kind of geometrical transformation. The text gives some reasoning as to why these transformations are linear mappings, and as to why they are affiliated with the matrices indicated. But a true understanding of these facts is not expected in this course. And so, in this lecture, I will summarize the results of section 3.3, so that you can focus on the calculations needed for these geometrical transformations. You will not be expected to prove that any of these transformations are linear mappings, to prove that the indicated matrices actually correspond to the affiliated transformation, or to graph any of the transformations. Again, the main goal of this section is simply to provide you with more examples of (and thus, more practice with) linear mappings.

Rotations through θ in $\mathbb{R}^2$

Definition: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is defined to be the transformation that rotates $\vec{x}$ counterclockwise through angle θ to the image $R_\theta(\vec{x})$. The standard matrix for R_θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Example: Imagine what happens to the point $(1,1)$ under various rotations. A rotation of $\pi/2$ would move $(1,1)$ from the first quadrant to the point $(-1,1)$ in the second quadrant. A rotation of π would send $(1,1)$ to the third quadrant point $(-1,-1)$. The smaller rotation of $-\pi/4$ would land $(1,1)$ on the the x-axis, on the point $(\sqrt{2},0)$. (Remember that the length of $\vec{x}$ will not change because of a rotation!) All of these facts can be verified using the standard matrix for a rotation:

$$R_{\pi/2}(1,1) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$R_{\pi}(1,1) = \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$R_{-\pi/4}(1,1) = \begin{bmatrix} \cos-\frac{\pi}{4} & -\sin-\frac{\pi}{4} \\ \sin-\frac{\pi}{4} & \cos-\frac{\pi}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Stretches and Shears

<u>Definition</u> For $t \in \mathbb{R}$, $t > 0$, a **stretch** by a factor of t in the x_i direction means to multiply the x_i term by t, but leave all other terms unchanged. Visually, we are pulling $\vec{x}$ in the x_i direction, but the amount of pulling depends on the distance of $\vec{x}$ from the origin (approximated by the x_i term). If $t < 1$, this is is sometimes referred to as a **shrink** instead of a stretch. The matrix for a stretch is obtained by replacing the "1" in the ii-th term of the identity matrix with t.

Example: In $\mathbb{R}^2$, the matrix for a stretch by a factor of 2 in the x_2 direction is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$,
while in $\mathbb{R}^4$, the matrix for a stretch by a factor of 2 in the x_2 direction is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

A shear is similar to a stretch, but instead of pulling the x_i term in the x_i direction, we pull the x_i term in the x_j direction, where $i \neq j$.

<u>Definition</u>: For $s \in \mathbb{R}$, a **shear** of x_i by a factor of s in the x_j direction means to "push" $\vec{x}$ in the x_i direction by sx_j (where $j \neq i$). Thus, the amount of shear applied to $\vec{x}$ depends both on s and on how far $\vec{x}$ is from the origin (which is approximated by x_j). The matrix for a shear is obtained by replacing the 0 in the ij-th term of the identity matrix with s.

Example: In $\mathbb{R}^2$, the matrix for a shear by a factor of $-3x_1$ in the x_2 direction is $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$. (In this example, $i = 2$ and $j = 1$, so we replace the 21-term in the identity matrix with -3.)
In $\mathbb{R}^3$, the matrix for a shear by a factor of $8x_3$ in the x_2 direction is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$.
This is the same as the function $S(x_1, x_2, x_3) = (x_1, x_2 + 8x_3, x_3)$.

An idea related to stretches and shears is a translation. A translation is when you move an object to a different location. Mathematically, this is achieved by adding a fixed vector to your vectors. (Example: $T(x_1, x_2) = (x_1, x_2) + (2, 3)$ would define a translation. Usually this is written as $T(x_1, x_2) = (x_1 + 2, x_2 + 3)$.) So why aren't we talking about translations in this section? Because they aren't linear mappings. I mention them to emphasize the difference between a translation like $T(x_1, x_2) = (x_1 + 2, x_2 + 3)$, which is not linear, and a stretch like $S_1(x_1, x_2) = (2x_1, x_2)$ or a shear like $S_2(x_1, x_2) = (x_1 + 3x_2, x_2)$ both of which are linear mappings.

Contractions and Dilations

While a stretch or shear only moves a point along one axis, a contraction/dilation stretches the point along all the axes simultaneously. Imagine dilating a circle of radius 1 to a circle of radius 5, or contracting a square with sides of length 10 to a square with sides of length 3. The shape will remain the same, but the size changes.

Definition: For $t \in \mathbb{R}$, $t > 1$, the **dilation** of $\vec{x}$ by a factor of t is the function $T(\vec{x}) = t\vec{x}$. If $0 < t < 1$, the function $T(\vec{x}) = t\vec{x}$ is called the **contraction** of $\vec{x}$ by a factor of t. As these are the same function, they have the same standard matrix, which is obtained by multiplying the identity matrix by t.

Example: A dilation by a factor of 3 in $\mathbb{R}^3$ has matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
A contraction by a factor of $\frac{2}{7}$ in $\mathbb{R}^2$ has matrix $\begin{bmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{bmatrix}$.

Reflections

The easiest reflection to consider is a reflection across a coordinate axis in $\mathbb{R}^2$. For example, if we go back to that most standard of points $(1, 1)$, then a reflection across the x_1 axis would send $(1, 1)$ to $(1, -1)$, while a reflection across the x_2 axis would send $(1, 1)$ to $(-1, 1)$. But we need not restrict ourselves to reflections across a coordinate axis. We can reflect across any line through the origin in $\mathbb{R}^2$. And more generally, we can reflect across any plane through the origin in $\mathbb{R}^3$, and through any hyperplane through the origin in a general $\mathbb{R}^n$. (The textbook limits itself to $\mathbb{R}^2$ and $\mathbb{R}^3$, and so I shall too from this point forward, but I thought I'd at least present the idea that our formula does generalize to $\mathbb{R}^n$.) The key is that all of these objects can be described by an equation of the form $\vec{n} \cdot \vec{x} = 0$. We can then use the normal vector $\vec{n}$ to describe the location of an arbitrary point $\vec{p}$ relative to the line $\vec{n} \cdot \vec{x} = 0$, and use that information to plot its new location after the reflection.

Definition: Let $\vec{n} \cdot \vec{x} = 0$ define a line (plane) through the origin in $\mathbb{R}^2$ ($\mathbb{R}^3$). A **reflection in the line/plane with normal vector** $\vec{n}$ will be denoted $\text{refl}_{\vec{n}}$, and we have that

$$\text{refl}_{\vec{n}}(\vec{p}) = \vec{p} - 2\text{proj}_{\vec{n}}(\vec{p})$$

Example: To determine the matrix of the reflection in the line $2x_1 + 3x_2 = 0$ in $\mathbb{R}^2$, we need to compute $\text{refl}_{\vec{n}}\vec{e}_1$ and $\text{refl}_{\vec{n}}\vec{e}_2$. To do this, the very first thing we need to do is read off the fact that $\vec{n} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ from our equation. Next, in order to compute the necessary $\text{refl}_{\vec{n}}$ values, we first need to compute the related $\text{proj}_{\vec{n}}$ values. To that end, we first note the following calculations:

$$||\vec{n}||^2 = 2^2 + 3^2 = 13 \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3$$

Then we see that $\text{proj}_{\vec{n}}\vec{e}_1 = \frac{\vec{e}_1 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{2}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/13 \\ 6/13 \end{bmatrix}$,
and $\text{proj}_{\vec{n}}\vec{e}_2 = \frac{\vec{e}_2 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{3}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6/13 \\ 9/13 \end{bmatrix}$.

And now we can compute that $\text{refl}_{\vec{n}}\vec{e}_1 = \vec{e}_1 - 2\text{proj}_{\vec{n}}\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 4/13 \\ 6/13 \end{bmatrix} = \begin{bmatrix} 5/13 \\ -12/13 \end{bmatrix}$,
and $\text{refl}_{\vec{n}}\vec{e}_2 = \vec{e}_2 - 2\text{proj}_{\vec{n}}\vec{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 6/13 \\ 9/13 \end{bmatrix} = \begin{bmatrix} -12/13 \\ -5/13 \end{bmatrix}$.

Thus, we have that $[\text{refl}_{\vec{n}}] = \begin{bmatrix} 5/13 & -12/13 \\ -12/13 & -5/13 \end{bmatrix}$.