Lecture 3j

Linear Mappings = Matrix Mappings

(pages 136-138)

Theorem 3.2.2 tells us that every matrix mapping is a linear mapping, but it turns out that every linear mapping can be described as a matrix mapping. Before we state this theorem, let's look at an example.

Example: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L(x_1, x_2, x_3) = (x_1 + x_3, x_2 + x_3)$. We saw in the previous lecture that L is a linear mapping. Now, let's look at the images of the standard basis vectors under L:

$$L(1,0,0) = (1,0) \qquad L(0,1,0) = (0,1) \qquad L(0,0,1) = (1,1)$$

But consider that, since L is a linear mapping, we have

$$\begin{aligned} L(x_1, x_2, x_3) &= L(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3) \\ &= x_1 L(\vec{e}_1) + x_2 L(\vec{e}_2) + x_3 L(\vec{e}_3) \\ &= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

So, we see that L is the same function as the matrix mapping f_A, where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Now, this has all been done for a specific L, but we can easily generalize to an arbitrary L, using the fact that any vector $\vec{v}$ can be written as the linear combination $v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$, and thus

$$\begin{aligned} L(\vec{v}) &= v_1 L(\vec{e}_1) + v_2 L(\vec{e}_2) + \cdots + v_n L(\vec{e}_n) \\ &= \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) & \cdots & L(\vec{e}_n) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \end{aligned}$$

Theorem 3.2.3: If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then L can be represented as a matrix mapping, with the corresponding $m \times n$ matrix $[L]$ given by

$$[L] = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) & \cdots & L(\vec{e}_n) \end{bmatrix}$$

$[L]$ is known as the **standard matrix** for the linear mapping L.

Example: The textbook notes that for any fixed vector $\vec{v} \in \mathbb{R}^n$, the function $\text{proj}_{\vec{v}}$ that sends $\vec{x} \in \mathbb{R}^n$ to $\text{proj}_{\vec{v}}\vec{x}$, is a linear mapping, and gives an example of how to find $[\text{proj}_{\vec{v}}]$. So I'll look at the related function $\text{perp}_{\vec{v}}$ (that sends $\vec{x} \in \mathbb{R}^n$ to $\text{perp}_{\vec{v}}\vec{x}$). Let $\vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. To find the standard matrix for $\text{perp}_{\vec{v}}$, we need to look at $\text{perp}_{\vec{v}}\vec{e}_1$ and $\text{perp}_{\vec{v}}\vec{e}_2$. Recall that

$$\text{perp}_{\vec{v}}\vec{e}_1 = \vec{e}_1 - \left(\tfrac{\vec{e}_1 \cdot \vec{v}}{||\vec{v}||^2}\right)\vec{v} \quad \text{and} \quad \text{perp}_{\vec{v}}\vec{e}_2 = \vec{e}_2 - \left(\tfrac{\vec{e}_2 \cdot \vec{v}}{||\vec{v}||^2}\right)\vec{v}$$

To calculate these, we can first note that

$$\vec{e}_1 \cdot \vec{v} = -3, \quad \vec{e}_2 \cdot \vec{v} = 2, \quad \text{and} \quad ||\vec{v}||^2 = (-3)^2 + (2)^2 = 13$$

Thus, we have that

$$\text{perp}_{\vec{v}}\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \left(\frac{-3}{13}\right)\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -9/13 \\ 6/13 \end{bmatrix} = \begin{bmatrix} 4/13 \\ 6/13 \end{bmatrix}$$

and

$$\text{perp}_{\vec{v}}\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \left(\frac{2}{13}\right)\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 6/13 \\ -4/13 \end{bmatrix} = \begin{bmatrix} 6/13 \\ 9/13 \end{bmatrix}$$

Thus, $[\text{perp}_{\vec{v}}] = \begin{bmatrix} 4/13 & 6/13 \\ 6/13 & 9/13 \end{bmatrix}$

It is important to note that only linear mappings satisfy the result of Theorem 3.2.3. Not every function can be described by a matrix.

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x_1, x_2) = ((x_1)^2, (x_2)^2)$. Then $f(1,0) = (1,0)$, and $f(0,1) = (0,1)$. If f was linear, then its standard matrix would be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But $f(2,2) = (4,4)$, while $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. So the function $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$ is not the same function as f. Of course, the function $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$ is simply the identity function on $\mathbb{R}^2$. That is, it is the function that sends $\vec{x}$ to $\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.