Lecture 3i

Linear Mappings

(pages 134-135)

In the previous lecture we noted that matrix mappings have two useful properties, specifically that they preserve addition and they preserve scalar multiplication. Any function with these two useful properties is known as a linear mapping.

Definition: A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear mapping** or **linear transformation** if for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ it satisfies the following properties:

(L1) $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$

(L2) $L(t\vec{x}) = tL(\vec{x})$.

Definition: A **linear operator** is a linear mapping whose domain and codomain are the same.

The "linear" in these definitions is the same as in "linear combinations", and in fact an alternate definition for a linear mapping is that it is a function such that $L(a\vec{x} + b\vec{y}) = aL(\vec{x}) + bL(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$. (A slightly different, but also equivalent, alternate definition is given in the textbook.) The idea being that a linear mapping should preserve linear combinations.

Example: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L(x_1, x_2, x_3) = (x_1 + x_3, x_2 + x_3)$. Then L is a linear mapping. To show this, we need to show that L preserves addition and that L preserves scalar multiplication. We show that L preserves addition as follows:

$$\begin{aligned}
L(\vec{x} + \vec{y}) &= L(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= ((x_1 + y_1) + (x_3 + y_3), (x_2 + y_2) + (x_3 + y_3)) \\
&= ((x_1 + x_3) + (y_1 + y_3), (x_2 + x_3) + (y_2 + y_3)) \\
&= (x_1 + x_3, x_2 + x_3) + (y_1 + y_3, y_2 + y_3) \\
&= L(\vec{x}) + L(\vec{y})
\end{aligned}$$

We show that L preserves scalar multiplication as follows:

$$\begin{aligned}
L(t\vec{x}) &= L(tx_1, tx_2, tx_3) \\
&= (tx_1 + tx_3, tx_2 + tx_3) \\
&= (t(x_1 + x_3), t(x_2 + x_3)) \\
&= t(x_1 + x_3, x_2 + x_3) \\
&= tL(\vec{x})
\end{aligned}$$

Example: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $L(x_1, x_2, x_3) = (x_1x_3, x_2x_3)$. Then L is not a linear transformation. To show this, consider the following:

$$L(1, 2, 3) = (3, 6) \text{ and } L(1, 2, 4) = (4, 8)$$

so

$$L(1,2,3) + L(1,2,4) = (3,6) + (4,8) = (7,14)$$

But

$$L((1,2,3) + (1,2,4)) = L(2,4,7) = (14,28) \neq L(1,2,3) + L(1,2,4)$$

This demonstrates that L does not preserve addition, and therefore L cannot be a linear transformation.

One counterexample of either preserving addition or preserving scalar multiplication is all that is required to show that a function is not a linear transformation, but it turns out that L also does not preserve scalar multiplication, so for the purposes of an example I will show this as well. Consider that $L(1,2,3) = (3,6)$, so $2L(1,2,3) = 2(3,6) = (6,12)$. But $L(2(1,2,3)) = L(2,4,6) = (12,24) \neq 2L(1,2,3)$, so we see that L does not preserve scalar multiplication.