Lecture 3h

Properties of Matrix Mappings

(pages 133-134)

Example: Let $A = \begin{bmatrix} 4 & 2 & -2 \\ 5 & -3 & -5 \end{bmatrix}$. Let's look at the images of the standard basis vectors $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, and $\vec{e}_3 = (0,0,1)$:

$$f_A(1,0,0) = \begin{bmatrix} 4 & 2 & -2 \\ 5 & -3 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$f_A(0,1,0) = \begin{bmatrix} 4 & 2 & -2 \\ 5 & -3 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$f_A(0,0,1) = \begin{bmatrix} 4 & 2 & -2 \\ 5 & -3 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

We notice that $f_A(1,0,0)$ is the first column of A, $f_A(0,1,0)$ is the second column of A, and $f_A(1,0,0)$ is the third column of A. So, we have

$$A = \begin{bmatrix} f_A(\vec{e}_1) & f_A(\vec{e}_2) & f_A(\vec{e}_3) \end{bmatrix}$$

And now consider $f_A(2,5,-7)$. We could use our definition of f_A and get

$$f_A(2,5,-7) = \begin{bmatrix} 4 & 2 & -2 \\ 5 & -3 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 8+10+14 \\ 10-15+35 \end{bmatrix} = \begin{bmatrix} 32 \\ 30 \end{bmatrix}$$

Or we could use block multiplication, and get

$$\begin{aligned} f_A(2,5,-7) &= \begin{bmatrix} f_A(\vec{e}_1) & f_A(\vec{e}_2) & f_A(\vec{e}_3) \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix} \\ &= [2f_A(\vec{e}_1) + 5f_A(\vec{e}_2) - 7f_A(\vec{e}_3)] \\ &= 2\begin{bmatrix} 4 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 2 \\ -3 \end{bmatrix} - 7\begin{bmatrix} -2 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 8+10+14 \\ 10-15+35 \end{bmatrix} \\ &= \begin{bmatrix} 32 \\ 30 \end{bmatrix} \end{aligned}$$

Right now I'm sure you can't imagine why you'd go through this longer process, but I did so to illustrate the following theorem.

Theorem 3.2.1: Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be the standard basis vectors of $\mathbb{R}^n$, let A be any $m \times n$ matrix, and let $f_A : \mathbb{R}^n \to \mathbb{R}^m$ be the corresponding matrix mapping. Then, for any vector $\vec{x} \in \mathbb{R}^n$, we have

$$f_A(\vec{x}) = x_1 f_A(\vec{e}_1) + x_2 f_A(\vec{e}_2) + \cdots + x_n f_A(\vec{e}_n)$$

The proof of Theorem 3.2.1 is similar to what I did in the example. That is, you note that $f_A(\vec{e}_k)$ is the k-th column of A, and then use block multiplication to compute $f_A(\vec{x})$. But Theorem 3.2.1 is just a special case of a more general property of matrix mappings.

Theorem 3.2.2: Let A be an $m \times n$ matrix, with corresponding matrix mapping $f_A : \mathbb{R}^n \to \mathbb{R}^m$. Then, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ and any $t \in \mathbb{R}$, we have
(L1) $f_A(\vec{x} + \vec{y}) = f_A(\vec{x}) + f_A(\vec{y})$
(L2) $f_A(t\vec{x}) = t f_A(\vec{x})$.

The proof of this theorem is an easy application of the properties of matrix multiplication, and as it is given in the textbook I won't bother reproducing it here. However, the fact that this theorem is easy to prove should not detract from its importance. In fact, you should take note of the fact that instead of using the standard numbering, we have labeled these properties "L1" and "L2," which clearly conveys a sense of importance. In fact, ANY function that satisfies property L1 is said to **preserve addition**, and any function that satisfies property L2 is said to **preserve scalar multiplication**.

Example: Let f_A be a matrix mapping such that $f_A(1, 0, 0) = (1, -2, -5)$, $f_A(0, 1, 0) = (0, 3, -5)$, and $f_A(0, 0, 1) = (9, -2, 0)$. Then we can compute $f_A(3, -1, 6)$, even though we don't know A, as follows:

$$\begin{array}{rcll} f_A(3,-1,6) & = & f_A(3(1,0,0) - (0,1,0) + 6(0,0,1)) & \\ & = & f_A(3(1,0,0)) - f_A(0,1,0) + f_A(6(0,0,1)) & \text{(using property L1)} \\ & = & 3f_A(1,0,0) - f_A(0,1,0) + 6f_A(0,0,1) & \text{(using property L2)} \\ & = & 3(1,-2,-5) - (0,3-5) + 6(9,-2,0) & \\ & = & (3-0+54, -2-3-12, -15+5+0) & \\ & = & (57,-17,-10) & \end{array}.$$

We could have skipped several of these steps by using Theorem 3.2.1, but doing the calculation this way illustrates the fact that Theorem 3.2.1 is a special case of Theorem 3.2.2.

Example: Suppose that $f_A(2, 0, 0) = (4, -6, -2)$, $f_A(0, 3, 0) = (-9, -27, 15)$, and $f_A(0, 0, 5) = (25, 5, -15)$. Then we can find A as follows:

$$\begin{array}{rcccccc} f_A(1,0,0) & = & \frac{1}{2}f_A(2,0,0) & = & \frac{1}{2}(4,-6,-2) & = & (2,-3,-1) \\ f_A(0,1,0) & = & \frac{1}{3}f_A(0,3,0) & = & \frac{1}{3}(-9,-27,15) & = & (-3,-9,5) \\ f_A(0,0,1) & = & \frac{1}{5}f_A(0,0,5) & = & \frac{1}{5}(25,5,-15) & = & (5,1,-3) \end{array}$$

We know that $A = \left[\begin{array}{ccc} f_A(\vec{e}_1) & f_A(\vec{e}_2) & f_A(\vec{e}_3) \end{array}\right]$, so we now see that

$$A = \left[\begin{array}{ccc} 2 & -3 & 5 \\ -3 & -9 & 1 \\ -1 & 5 & -3 \end{array}\right]$$