Lecture 3g

Matrix Mappings

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We want to start looking at functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, but before we do so let's do a quick refresher of the various terms used when talking about functions.

Definition: A **function** $f$ is a rule that assigns to every element $x$ of a set called the **domain** a unique value $y$ in another set called the **codomain**.

**Examples:** Let the domain and codomain be $\mathbb{R}$. Then the rule "$x$ goes to $y = x^2$" is a function, the rule "$x$ goes to $y = x + 6$" is a function, and the rule "$x$ goes to $y = 7$" is a function. However, the rule "$x$ goes to $y$ such that $y^2 = x$" is not a function. The first problem is that this rule does not assign a $y$ to every real number $x$. For example, there would be no $y$ value for $x = -1$. If we restrict the domain to the non-negative reals, this rule is still not a function, because the rule must identify a UNIQUE value $y$, but the rule I gave would assign both $y = 2$ and $y = -2$ to $x = 4$. If we restrict both the domain and codomain to the non-negative reals, then the rule "$x$ goes to $y$ such that $y^2 = x$" is a function.

The goal in this lecture is not to do an in depth study of functions on $\mathbb{R}$, but I did want to emphasize that not everything is a function, and that you need to pay attention to what the stated domain and codomain are.

Notation: If $f$ is a function with domain $U$ and codomain $V$, then we say that $f$ maps $U$ to $V$, and denote this by $f : U \to V$.

Notation: We say that a function $f$ maps $x$ to $y$, or that $y$ is the image of $x$ under $f$, and we write $f(x) = y$.

**Example:** Consider the function $f : \mathbb{R}^4 \to \mathbb{R}^2$, such that $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$. Then

$$f\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

**Example:** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. We first note that the product $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in fact a $3 \times 1$ matrix, and thus is the same as an element of $\mathbb{R}^3$. And we have that

$$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The idea of using a matrix product to define a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ turns out to be a very important idea, and so we make the following definition.

Definition: For any $m \times n$ matrix $A$, we define a function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ called the **matrix mapping** corresponding to $A$ by

$$f_A(\vec{x}) = A\vec{x} \qquad \text{for any } \vec{x} \in \mathbb{R}^n.$$

Notation: Mostly just for purposes of typesetting, we sometimes write the vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as the point $(x_1, x_2, \ldots, x_n)$.

**Example**: Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then the domain of $f_A$ is $\mathbb{R}^4$ (since $A$ has 4 columns) and the codomain of $f_A$ is $\mathbb{R}^2$ (since $A$ has 2 rows). And we have that

$$f_A(1,2,3,4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = (2,4)$$

$$f_A(4,-2,9,-1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = (-2,-1)$$

$$f_A(1,0,0,0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (0,0)$$

$$f_A(0,1,0,0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1,0)$$

Notice that $f_A$ is the same as the function $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$ I used in an earlier example. We can prove this by noting that

$$f_A(x_1,x_2,x_3,x_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

for all $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$.