Solution to Practice 3e

B5(a) $AB = \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 4 & 5 \end{bmatrix} =$

$$\begin{bmatrix} (2)(3)+(1)(4) & (2)(-4)+(1)(5) \\ (-1)(3)+(-2)(4) & (-1)(-4)+(-2)(5) \\ (4)(3)+(-3)(4) & (4)(-4)+(-3)(5) \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -11 & -6 \\ 0 & -31 \end{bmatrix}$$

B5(b) BA is not defined, because the number of columns in B does not equal the number of rows in A.

B5(c) $AC = \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -2 & 5 & -2 \\ -2 & 1 & 0 \end{bmatrix} =$

$$\begin{bmatrix} (2)(-2)+(1)(-2) & (2)(5)+(1)(1) & (2)(-2)+(1)(0) \\ (-1)(-2)+(-2)(-2) & (-1)(5)+(-2)(1) & (-1)(-2)+(-2)(0) \\ (4)(-2)+(-3)(-2) & (4)(5)+(-3)(1) & (4)(-2)+(-3)(0) \end{bmatrix} = \begin{bmatrix} -6 & 11 & -4 \\ 6 & -7 & 2 \\ -2 & 17 & -8 \end{bmatrix}$$

B5(d) DC is not defined, because the number of columns in D does not equal the number of rows in C.

B5(e) $DA = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 2 & 1 \\ -3 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 4 & -3 \end{bmatrix} =$

$$\begin{bmatrix} (1)(2)+(3)(-1)+(-5)(4) & (1)(1)+(3)(-2)+(-5)(-3) \\ (0)(2)+(2)(-1)+(1)(4) & (0)(1)+(2)(-2)+(1)(-3) \\ (-3)(2)+(2)(-1)+(1)(4) & (-3)(1)+(2)(-2)+(1)(-3) \\ (1)(2)+(1)(-1)+(-1)(4) & (1)(1)+(1)(-2)+(-1)(-3) \end{bmatrix} = \begin{bmatrix} -21 & 10 \\ 2 & -7 \\ -4 & -10 \\ -3 & 2 \end{bmatrix}$$

B5(f) $D^T = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 3 & 2 & 2 & 1 \\ -5 & 1 & 1 & -1 \end{bmatrix}$, so $CD^T = \begin{bmatrix} -2 & 5 & -2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 3 & 2 & 2 & 1 \\ -5 & 1 & 1 & -1 \end{bmatrix} =$

$$\begin{bmatrix} 23 & 8 & 14 & 5 \\ 1 & 2 & 8 & -1 \end{bmatrix}$$

B5(g) $CA = \begin{bmatrix} -2 & 5 & -2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 4 & -3 \end{bmatrix} =$

$$\begin{bmatrix} (-2)(2)+(5)(-1)+(-2)(4) & (-2)(1)+(5)(-2)+(-2)(-3) \\ (-2)(2)+(1)(-1)+(0)(4) & (-2)(1)+(1)(-2)+(0)(-3) \end{bmatrix} = \begin{bmatrix} -17 & -6 \\ -5 & -4 \end{bmatrix},$$

so

$$B(CA) = \begin{bmatrix} 3 & -4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -17 & -6 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} (3)(-17)+(-4)(-5) & (3)(-6)+(-4)(-4) \\ (4)(-17)+(5)(-5) & (4)(-6)+(5)(-4) \end{bmatrix} =$$

$\begin{bmatrix} -31 & -2 \\ -93 & -44 \end{bmatrix}$. And

$BC = \begin{bmatrix} 3 & -4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -2 & 5 & -2 \\ -2 & 1 & 0 \end{bmatrix} =$
$\begin{bmatrix} (3)(-2)+(-4)(-2) & (3)(5)+(-4)(1) & (3)(-2)+(-4)(0) \\ (4)(-2)+(5)(-2) & (4)(5)+(5)(1) & (4)(-2)+(5)(0) \end{bmatrix} =$
$\begin{bmatrix} 2 & 11 & -6 \\ -18 & 25 & -8 \end{bmatrix}$, so

$(BC)A = \begin{bmatrix} 2 & 11 & -6 \\ -18 & 25 & -8 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 4 & -3 \end{bmatrix} =$
$\begin{bmatrix} (2)(2)+(11)(-1)+(-6)(4) & (2)(1)+(11)(-2)+(-6)(-3) \\ (-18)(2)+(25)(-1)+(-8)(4) & (-18)(1)+(25)(-2)+(-8)(-3) \end{bmatrix} =$
$\begin{bmatrix} -31 & -2 \\ -93 & -44 \end{bmatrix} = B(CA)$

B5(h) From (a), we know that $AB = \begin{bmatrix} 10 & -3 \\ -11 & -6 \\ 0 & -31 \end{bmatrix}$, so $(AB)^T = \begin{bmatrix} 10 & -11 & 0 \\ -3 & -6 & -31 \end{bmatrix}$.
We also have that $B^T = \begin{bmatrix} 3 & 4 \\ -4 & 5 \end{bmatrix}$ and $A^T = \begin{bmatrix} 2 & -1 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, so

$B^T A^T = \begin{bmatrix} 3 & 4 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 \\ 1 & -2 & -3 \end{bmatrix} =$
$\begin{bmatrix} (3)(2)+(4)(1) & (3)(-1)+(4)(-2) & (3)(4)+(4)(-3) \\ (-4)(2)+(5)(1) & (-4)(-1)+(5)(-2) & (-4)(4)+(5)(-3) \end{bmatrix} = \begin{bmatrix} 10 & -11 & 0 \\ -3 & -6 & -31 \end{bmatrix} =$
$(AB)^T$

B5(i) $(D^T)^T C^T = (CD^T)^T =$ (from(f)) $\begin{bmatrix} 23 & 8 & 14 & 5 \\ 1 & 2 & 8 & -1 \end{bmatrix}^T = \begin{bmatrix} 23 & 1 \\ 8 & 2 \\ 14 & 8 \\ 5 & -1 \end{bmatrix}$

B8(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \end{bmatrix} = \begin{bmatrix} (3)(-2) & (3)(4) \\ (2)(-2) & (2)(4) \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ -4 & 8 \end{bmatrix}$

B8(b) $\begin{bmatrix} -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} (-2)(3)+(4)(2) \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$

B8(c) $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} (2)(-3) & (2)(1) & (2)(2) \\ (1)(-3) & (1)(1) & (1)(2) \\ (5)(-3) & (5)(1) & (5)(2) \end{bmatrix} = \begin{bmatrix} -6 & 2 & 4 \\ -3 & 1 & 2 \\ -15 & 5 & 10 \end{bmatrix}$

B8(d) $\begin{bmatrix} -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} (-3)(2) + (1)(1) + (2)(5) \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$

D2 First, let's look at $I_m A$: $(I_m A)_{ij} = \sum_{k=1}^{n} (I_m)_{ik} A_{kj}$. But $(I_m)_{ik} = 0$ for $i \neq k$, so $\sum_{k=1}^{n} (I_m)_{ik} A_{kj} = (I_m)_{ii} A_{ij}$. Moreover, $(I_m)_{ii} = 1$, so we see that $(I_m A)_{ij} = A_{ij}$. Since this is true for all i and j, we see that $A_m I = A$.

The proof that $AI_n = A$ is similar: $(AI_n)_{ij} = \sum_{k=1}^{n} (A)_{ik} (I_n)_{kj}$. But $(I_n)_{kj} = 0$ for $k \neq j$, so $\sum_{k=1}^{n} (A)_{ik} (I_n)_{kj} = (A)_{ij} (I_n)_{jj}$. And since $(I_n)_{jj} = 1$, we see that $(AI_n)_{ij} = (A)_{ij}$. As this is true for all i and j, we see that $AI_n = A$.

D4(a) There are many possible answers to this. My favourite is $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

D4(b) The key to this question is to notice that $(A-B)^2 = (A-B)(A-B) = A^2 - AB - BA + B^2$ (using the distributive property: Theorem 3, (1)). So, if we want $A^2 - AB - BA + B^2 = O_{2,2}$, then we want $(A-B)^2 = O_{2,2}$. Now, this would be obvious if $A - B = O_{2,2}$, but we aren't allowed to let $A = B$. Luckily, in part (a), we found a non-zero matrix whose square is the zero matrix. Using my matrix from (a), I need to find non-zero matrices A and B such that $A - B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. So, I choose $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$.