Solution to Practice 3d

B2(a) $\begin{bmatrix} -3 & 2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 2 & -3 & -1 \end{bmatrix} =$

$\begin{bmatrix} (-3)(3)+(2)(2) & (-3)(1)+(2)(-3) & (-3)(-2)+(2)(-1) \\ (5)(3)+(1)(2) & (5)(1)+(1)(-3) & (5)(-2)+(1)(-1) \end{bmatrix} =$

$\begin{bmatrix} 5 & -9 & 4 \\ 17 & 2 & -11 \end{bmatrix}$

B2(b) $\begin{bmatrix} 0 & 3 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ 2 & -1 \\ 5 & 0 \end{bmatrix} =$

$\begin{bmatrix} (0)(7)+(3)(2)+(-1)(5) & (0)(-3)+(3)(-1)+(-1)(0) \\ (-1)(7)+(2)(2)+(-1)(5) & (-1)(-3)+(2)(-1)+(-1)(0) \\ (1)(7)+(1)(2)+(3)(5) & (1)(-3)+(1)(-1)+(3)(0) \end{bmatrix} =$

$\begin{bmatrix} 1 & -3 \\ -8 & 1 \\ 24 & -4 \end{bmatrix}$

B2(c) $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ -2 & 1 & -2 & 3 \end{bmatrix} =$

$\begin{bmatrix} (3)(3)+(-1)(-2) & (3)(1)+(-1)(1) & (3)(-2)+(-1)(-2) & (3)(2)+(-1)(3) \\ (2)(3)+(4)(-2) & (2)(1)+(4)(1) & (2)(-2)+(4)(-2) & (2)(2)+(4)(3) \\ (2)(3)+(7)(-2) & (2)(1)+(7)(1) & (2)(-2)+(7)(-2) & (2)(2)+(7)(3) \end{bmatrix} =$

$\begin{bmatrix} 11 & 2 & -4 & 3 \\ -2 & 6 & -12 & 16 \\ -8 & 9 & -18 & 25 \end{bmatrix}$

B2(d) This product does not exist, because the number of columns in $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}$ is not the same as the number of rows in $\begin{bmatrix} -4 & 1 & 5 \\ 6 & 3 & -1 \end{bmatrix}$.

B3(a)

$AB = \begin{bmatrix} (3)(4)+(5)(1) & (3)(2)+(5)(-6) & (3)(-1)+(5)(8) \\ (-2)(4)+(-1)(1) & (-2)(2)+(-1)(-6) & (-2)(-1)+(-1)(8) \end{bmatrix} = \begin{bmatrix} 17 & -24 & 37 \\ -9 & 2 & -6 \end{bmatrix}$

$AC = \begin{bmatrix} (3)(-2)+(5)(2) & (3)(1)+(5)(2) & (3)(4)+(5)(-3) \\ (-2)(-2)+(-1)(2) & (-2)(1)+(-1)(2) & (-2)(4)+(-1)(-3) \end{bmatrix} = \begin{bmatrix} 4 & 13 & -3 \\ 2 & -4 & -5 \end{bmatrix}$

So $AB + AC = \begin{bmatrix} 17+4 & -24+13 & 37-3 \\ -9+2 & 2-4 & -6-5 \end{bmatrix} = \begin{bmatrix} 21 & -11 & 34 \\ -7 & -2 & -11 \end{bmatrix}$

$B+C = \begin{bmatrix} 4-2 & 2+1 & -1+4 \\ 1+2 & -6+2 & 8-3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & -4 & 5 \end{bmatrix}$

So $A(B{+}C) = \begin{bmatrix} (3)(2)+(5)(3) & (3)(3)+(5)(-4) & (3)(3)+(5)(5) \\ (-2)(2)+(-1)(3) & (-2)(3)+(-1)(-4) & (-2)(3)+(-1)(5) \end{bmatrix} =$
$\begin{bmatrix} 21 & -11 & 34 \\ -7 & -2 & -11 \end{bmatrix} = AB + AC.$

$3B = \begin{bmatrix} 3(4) & 3(2) & 3(-1) \\ 3(1) & 3(-6) & 3(8) \end{bmatrix} = \begin{bmatrix} 12 & 6 & -3 \\ 3 & -18 & 24 \end{bmatrix}$

so $A(3B) = \begin{bmatrix} (3)(12)+(5)(3) & (3)(6)+(5)(-18) & (3)(-3)+(5)(24) \\ (-2)(12)+(-1)(3) & (-2)(6)+(-1)(-18) & (-2)(-3)+(-1)(24) \end{bmatrix} =$
$\begin{bmatrix} 51 & -72 & 111 \\ -27 & 6 & -18 \end{bmatrix} = 3\begin{bmatrix} 17 & -24 & 37 \\ -9 & 2 & -6 \end{bmatrix} = 3(AB).$

B3(b)

$AB = \begin{bmatrix} (3)(2)+(4)(1) & (3)(-3)+(4)(5) \\ (1)(2)+(0)(1) & (1)(-3)+(0)(5) \\ (-3)(2)+(5)(1) & (-3)(-3)+(5)(5) \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 2 & -3 \\ -1 & 34 \end{bmatrix}$

$AC = \begin{bmatrix} (3)(-4)+(4)(3) & (3)(-6)+(4)(1) \\ (1)(-4)+(0)(3) & (1)(-6)+(0)(1) \\ (-3)(-4)+(5)(3) & (-3)(-6)+(5)(1) \end{bmatrix} = \begin{bmatrix} 0 & -14 \\ -4 & -6 \\ 27 & 23 \end{bmatrix}$

so $AB + AB = \begin{bmatrix} 10 & -3 \\ -2 & -9 \\ 26 & 57 \end{bmatrix}$

$B+C = \begin{bmatrix} 2-4 & -3-6 \\ 1+3 & 5+1 \end{bmatrix} = \begin{bmatrix} -2 & -9 \\ 4 & 6 \end{bmatrix}$

So $A(B+C) = \begin{bmatrix} (3)(-2)+(4)(4) & (3)(-9)+(4)(6) \\ (1)(-2)+(0)(4) & (1)(-9)+(0)(6) \\ (-3)(-2)+(5)(4) & (-3)(-9)+(5)(6) \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -2 & -9 \\ 26 & 57 \end{bmatrix} =$
$AB + AC$

$3B = \begin{bmatrix} 3(2) & 3(-3) \\ 3(1) & 3(5) \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 3 & 15 \end{bmatrix}$

so $A(3B) = \begin{bmatrix} (3)(6)+(4)(3) & (3)(-9)+(4)(15) \\ (1)(6)+(0)(3) & (1)(-9)+(0)(15) \\ (-3)(6)+(5)(3) & (-3)(-9)+(5)(15) \end{bmatrix} = \begin{bmatrix} 30 & 33 \\ 6 & -9 \\ -3 & 102 \end{bmatrix} =$
$3\begin{bmatrix} 10 & 11 \\ 2 & -3 \\ -1 & 34 \end{bmatrix} = 3(AB).$

D3 Recall that our definition of matrix multiplication gives us $(BA)_{ij} = \vec{b}_i \cdot \vec{a}_j$, where the $\vec{b}_i$ are the rows of B, while the $\vec{a}_j$ are the columns of A. Well, suppose we replace B with A^T. Then $\vec{b}_i^T$ is the i-th row of A^T. But since the i-th row of A^T is the transpose of the i-th column of A, we have that $\vec{b}_i^T = \vec{a}_i$. As such, we have that $(A^T A)_{ij} = \vec{a}_i \cdot \vec{a}_j$. The find the formula for $(AA^T)_{ij}$, let $\vec{\alpha}_i$ be the i-th row of A, and let $\vec{b}_j$ be the j-th column of A^T. Then $\vec{b}_j$ is the transpose of the j-th row of A. That is, $\vec{b}_j = \vec{\alpha}_j^T$. So $(AA^T)_{ij} = \vec{\alpha}_i^T \cdot \vec{\alpha}_j^T$.

Summarizing these results, we see that the ij-th entry of $A^T A$ is the dot product of the i-th column of A with the j-th column of A, while the ij-th entry of AA^T is the dot product of the i-th row of A with the j-th row of A.

Now, lets look at what happens if $A^T A$ is the zero matrix. Then all of those dot products are 0. Specifically, let's look at the diagonal entries $(A^T A)_{ii}$. If $(A^T A)_{ii} = 0$, then $\vec{a}_i \cdot \vec{a}_i = 0$. But for any vector, we know $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$. This means that $\vec{a}_i = \vec{0}$, for all i. Since all the columns of A are the zero vector, we know that A must be the zero matrix.

Similarly, if AA^T is the zero matrix, then $(AA^T)_{ii} = 0$ for all i, which means that $\vec{\alpha}_i \cdot \vec{\alpha}_i = 0$ for all i, which means that $\vec{\alpha}_i = \vec{0}$ for all i. And so we see that all the rows of A are the zero vector, which means that A is the zero matrix.