Lecture 3f

Block Multiplication

(pages 127-128)

In the build up to defining the identity matrix, I had you look at making a matrix by lining up all the products $A\vec{e}_1\ A\vec{e}_2\ \cdots\ A\vec{e}_n$. This is an example of defining a matrix product (in this case, AI) by **block multiplication**. The idea is that, if you are willing to be flexible about the use of brackets in your matrix and vector notation, then instead of thinking of matrix multiplication as one giant matrix times another giant matrix, each matrix can be broken into small blocks, and we can do our calculations with these small blocks. Mind you, ideas are great, but there are some details to be worked out yet! The simplest form of block multiplication is similar to what we used to see $AI = A$.

Notice: Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix. Let $\vec{b}_i$ be the i-th column of B, so that $B = [\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_p]$. Then $AB = [A\vec{b}_1\ A\vec{b}_2\ \cdots\ A\vec{b}_p]$.

Example: Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 4 \\ 2 & -1 \end{bmatrix}$, and let $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$. Then:

$$A\vec{w} = \begin{bmatrix} (1)(1) + (-3)(2) \\ (0)(1) + (4)(2) \\ (2)(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix},$$

$$A\vec{x} = \begin{bmatrix} (1)(0) + (-3)(-1) \\ (0)(0) + (4)(-1) \\ (2)(0) + (-1)(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix},$$

$$A\vec{y} = \begin{bmatrix} (1)(3) + (-3)(8) \\ (0)(3) + (4)(8) \\ (2)(3) + (-1)(8) \end{bmatrix} = \begin{bmatrix} -21 \\ 32 \\ -2 \end{bmatrix}, \text{ and}$$

$$A\vec{z} = \begin{bmatrix} (1)(7) + (-3)(0) \\ (0)(7) + (4)(0) \\ (2)(7) + (-1)(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 14 \end{bmatrix}.$$

From this, we see that $A\begin{bmatrix} 1 & 0 & 3 & 7 \\ 2 & -1 & 8 & 0 \end{bmatrix} = \begin{bmatrix} A\vec{w} & A\vec{x} & A\vec{y} & A\vec{z} \end{bmatrix} =$
$\begin{bmatrix} -5 & 3 & -21 & 7 \\ 8 & -4 & 32 & 0 \\ 0 & 1 & -2 & 14 \end{bmatrix}$

Another common reason to use block multiplication is if you have matrices with lots of zeros. Not that these would be particularly difficult to multiply anyway, but if you can partition the matrices so that you have blocks of all zeros, things can go a bit quicker. Of course, first we need to notice the following:

<u>Notice</u>: Let A be an $m \times n$ matrix. Then $O_{p,m}A = O_{p,n}$ and $AO_{n,p} = O_{m,p}$ for any p.

Example: Calculate $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 0 & -9 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

Instead of just diving in, we can instead partition our matrices as follows:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 4 & 0 & -9 & 8 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

So, if we let $A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 & 3 \\ -9 & 8 \end{bmatrix}$, $B_{11} = \begin{bmatrix} 1 & 2 \\ -9 & 0 \end{bmatrix}$, $B_{12} =$ $B_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $B_{22} = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}$, then we have

$$AB = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{21} + A_{12}B_{22} \end{bmatrix}$$

(The idea is that we can treat the blocks as if they were individual entries. This only works if you set up your blocks so that all the matrix products are defined.)

But since B_{12} and B_{21} are the zero matrix, we have $A_{12}B_{21} = O_{2,2}$ and $A_{11}B_{21} = O_{2,2}$. Thus, $AB = \begin{bmatrix} A_{11}B_{11} & A_{12}B_{22} \end{bmatrix}$. And so we need only compute:

$$A_{11}B_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -9 & 0 \end{bmatrix} = \begin{bmatrix} -17 & 2 \\ 4 & 8 \end{bmatrix}, \text{ and}$$

$$A_{12}B_{22} = \begin{bmatrix} 1 & 3 \\ -9 & 8 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 18 \\ 0 & 13 \end{bmatrix}.$$

So $AB = \begin{bmatrix} -17 & 2 & 0 & 18 \\ 4 & 8 & 0 & 13 \end{bmatrix}$.

In general, this full blown version of block multiplication is not something I will require you to do. But it is useful when proving things, so I'll be using it from time to time in the lectures.