Lecture 3c

The Transpose of a Matrix

(pages 120-121)

As we start to explore the ways that a matrix is different from a giant vector, we begin in what might seem an unusual place.

Definition: Let A be an $m \times n$ matrix. Then the transpose of A is the $n \times m$ matrix A^T whose ij-th entry is the ji-th entry of A. That is,

$$(A^T)_{ij} = (A)_{ji}$$

Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

One of the more interesting things about the transpose is that it is an operation on a matrix that changes its size. As such, one can not in general look at $A + A^T$, or have that $A = A^T$. (Such things are possible if A is a square matrix, of course.) But there are some facts that we do know about the transpose:

Theorem 3.1.2: For any matrices A and B and scalar $s \in \mathbb{R}$, we have

(1) $(A^T)^T = A$
(2) $(A + B)^T = A^T + B^T$
(3) $(sA)^T = sA^T$

Example: Let $A = \begin{bmatrix} -3 & 0 \\ 9 & 2 \\ -2 & 2 \\ 7 & -1 \end{bmatrix}$. Then $A^T = \begin{bmatrix} -3 & 9 & -2 & 7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$, and $(A^T)^T = \begin{bmatrix} -3 & 0 \\ 9 & 2 \\ -2 & 2 \\ 7 & -1 \end{bmatrix} = A$. Moreover, we see that

$$(5A)^T = \begin{bmatrix} -15 & 0 \\ 45 & 10 \\ -10 & 10 \\ 35 & -5 \end{bmatrix}^T = \begin{bmatrix} -15 & 45 & -10 & 35 \\ 0 & 10 & 10 & -5 \end{bmatrix} = 5 \begin{bmatrix} -3 & 9 & -2 & 7 \\ 0 & 2 & 2 & -1 \end{bmatrix} = 5A^T$$

and if $B = \begin{bmatrix} 4 & 1 \\ -5 & -3 \\ 0 & 2 \\ -9 & 6 \end{bmatrix}$, then $B^T = \begin{bmatrix} 4 & -5 & 0 & 9 \\ 1 & -3 & 2 & 6 \end{bmatrix}$, and we see that

$$A + B = \begin{bmatrix} -3 & 0 \\ 9 & 2 \\ -2 & 2 \\ 7 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -5 & -3 \\ 0 & 2 \\ -9 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ -2 & 4 \\ -2 & 5 \end{bmatrix}$$

and

$$A^T + B^T = \begin{bmatrix} -3 & 9 & -2 & 7 \\ 0 & 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -5 & 0 & 9 \\ 1 & -3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 & -2 \\ 1 & -1 & 4 & 5 \end{bmatrix} = (A+B)^T$$