Lecture 3b

Span and Linear Independence

(pages 118-120)

So far we have defined matrices in a very similar way to the way we defined vectors in Chapter 1, so it should come as no surprise that we can also define linear combinations, spans, and linear independence as well.

Definition: Let $\mathcal{B} = \{A_1, \ldots, A_k\}$ be a set of $m \times n$ matrices, and let $t_1, \ldots, t_k$ be real scalars. Then $t_1A_1 + \cdots + t_kA_k$ is a linear combination of the matrices in $\mathcal{B}$.

Definition: Let $\mathcal{B} = \{A_1, \ldots, A_k\}$ be a set of $m \times n$ matrices. Then the **span** of $\mathcal{B}$ is defined as

$$\text{Span}\mathcal{B} = \{t_1A_1 + \cdots + t_kA_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$$

That is, Span$\mathcal{B}$ is the set of all linear combinations of the matrices in $\mathcal{B}$.

Definition: Let $\mathcal{B} = \{A_1, \ldots, A_k\}$ be a set of $m \times n$ matrices. Then $\mathcal{B}$ is said to be **linearly independent** if the only solution to the equation

$$t_1A_1 + \cdots + t_kA_k = O_{m,n}$$

is the trivial solution $t_1 = \cdots = t_k = 0$. Otherwise, $\mathcal{B}$ is said to be **linearly dependent**.

All of these definitions are exactly the same as their counterparts for vectors, except with an $m \times n$ matrix substituted for a vector in $\mathbb{R}^n$. Where things can become confusing is if you attempt to determine if a matrix is in Span$\mathcal{B}$, or if $\mathcal{B}$ is linearly independent. With vectors, we would create a system of linear equations, turn those into a matrix, and then row reduce. But wait–now are we supposed to make a matrix with matrix entries!?! NO! Instead, as you will see in the following examples, we will end up with a completely regular system of linear equations. Which we will then solve using matrices. I apologize if your head feels like its going to explode, but hopefully the following examples will make everything clear.

Example: Determine if $\begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} \right\}$.

To do this, we need to see if there are scalars t_1 and t_2 such that

$$t_1 \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix} + t_2 \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$$

Performing the operation on the left side, we see that we need

$$\begin{bmatrix} 2t_1 + 4t_2 & 4t_2 \\ 3t_1 - 9t_2 & -5t_1 + 8t_2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$$

By the definition of equality, that means that we need t_1 and t_2 to be solutions to all of the following equations:

$$\begin{array}{rcrcr} 2t_1 & + & 4t_2 & = & 6 \\ & & 4t_2 & = & 4 \\ 3t_1 & - & 9t_2 & = & 2 \\ -5t_1 & + & 8t_2 & = & 3 \end{array}$$

This is a system of linear equations! We can solve it by row reducing its augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 0 & 4 & 4 \\ 3 & -9 & 2 \\ -5 & 8 & 3 \end{array}\right] \begin{array}{l} (1/2)R_1 \\ (1/4)R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & -9 & 2 \\ -5 & 8 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 5R_1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -15 & -7 \\ 0 & 18 & 18 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 15R_2 \\ R_4 - 18R_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

Since the third row in our REF matrix is a bad row, we see that the system has no solutions. This means that there are no t_1 and t_2 such that

$$t_1 \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix} + t_2 \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$$

and thus, that $\begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$ is NOT in the span of $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ -9 & 8 \end{bmatrix} \right\}$.

Example: Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

To do this, we need to see if there are scalars t_1, t_2, t_3, and t_4 such that

$$t_1 \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + t_3 \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} + t_4 \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Performing the operation on the left side, we see that we need

$$\begin{bmatrix} t_1 - 2t_3 + 2t_4 & t_1 + 3t_2 + 4t_3 + 2t_4 \\ 2t_1 + 4t_2 - 4t_3 - 4t_4 & 2t_1 - 3t_2 - 5t_3 + 3t_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

By the definition of equality, this means we are looking for solutions to the following system of linear equations:

$$\begin{array}{rcrcrcrcr} t_1 & & & - & 2t_3 & + & 2t_4 & = & -1 \\ t_1 & + & 3t_2 & + & 4t_3 & + & 2t_4 & = & 2 \\ 2t_1 & + & 4t_2 & - & 4t_3 & - & 4t_4 & = & 2 \\ 2t_1 & - & 3t_2 & - & 5t_3 & + & 3t_4 & = & 1 \end{array}$$

We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 1 & 3 & 4 & 2 & 2 \\ 2 & 4 & -4 & -4 & 2 \\ 2 & -3 & -5 & 3 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 3 & 6 & 0 & 3 \\ 0 & 4 & 0 & -8 & 4 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{c} \\ (1/3)R_2 \\ (1/4)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 + 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] \begin{array}{c} \\ \\ (-1/2)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - 5R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -6 & 6 \end{array}\right] \begin{array}{c} \\ \\ \\ (-1/6)R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 - 2R_4 \\ \\ R_3 - R_4 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 + 2R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

We see from the RREF matrix that $t_1 = 3$, $t_2 = -1$, $t_3 = 1$, $t_4 = -1$ is a solution to our system. This means that

$$3\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

and thus, that $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ IS in the span of $\left\{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix}\right\}$.

Example: Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ is linearly independent.

To do this, we need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} + t_2 \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix} + t_3 \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix} t_1 + 8t_3 & 3t_1 - 2t_2 + 6t_3 \\ -t_1 + t_2 - 5t_3 & -3t_1 + 5t_2 - t_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and this is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & & & + & 8t_3 & = & 0 \\ 3t_1 & - & 2t_2 & + & 6t_3 & = & 0 \\ -t_1 & + & t_2 & - & 5t_3 & = & 0 \\ -3t_1 & + & 5t_2 & - & t_3 & = & 0 \end{array}$$

We solve this system by row reducing the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 8 \\ 3 & -2 & 6 \\ -1 & 1 & 5 \\ -3 & 5 & 1 \end{bmatrix} \begin{array}{c} \\ R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & -2 & 18 \\ 0 & 1 & 13 \\ 0 & 5 & 25 \end{bmatrix} \begin{array}{c} \\ (-1/2)R_2 \\ \\ (1/5)R_4 \end{array}$$
$$\sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 1 & 13 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 14 \end{bmatrix} \begin{array}{c} \\ \\ \\ R_4 - (14/22)R_3 \end{array}$$
$$\sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 0 \end{bmatrix}.$$

This final matrix is in row echelon form, and so we see that the rank of the coefficient matrix is 3. Since this is the same as the number of variables, there are no parameters in the general solution to our homogeneous system. This means that there is only one solution to the system, and we know that this must be the trivial solution. And this means that the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ IS linearly independent.

Example: Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} \right\}$ is linearly independent.

To do this, we need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} + t_2 \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix} + t_3 \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix} t_1 - t_2 + 2t_3 & t_1 + 2t_2 + 11t_3 \\ 3t_1 - 8t_2 - 9t_3 & -t_1 + 3t_2 + 4t_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and this is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & - & t_2 & + & 2t_3 & = & 0 \\ t_1 & + & 2t_2 & + & 11t_3 & = & 0 \\ 3t_1 & - & 8t_2 & - & 9t_3 & = & 0 \\ -t_1 & + & 3t_2 & + & 4t_3 & = & 0 \end{array}$$

We solve this system by row reducing the coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 11 \\ 3 & -8 & -9 \\ -1 & 3 & 4 \end{bmatrix} \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 + R_1 \end{array} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 9 \\ 0 & -5 & -15 \\ 0 & 2 & 6 \end{bmatrix} \begin{array}{c} \\ (1/3)R_2 \\ (-1/5)R_3 \\ (1/2)R_4 \end{array}$$
$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - R_3 \end{array} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This final matrix is in row echelon form, so we see that the rank of the coefficient matrix is 2. Since the number of variables in the system is 3, this means that there are 3-2=1 parameters in the general solution to the system. Thus, $t_1 = t_2 = t_3 = 0$ is not the only solution to our equation, and this means that $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} \right\}$ is linearly dependent. (That is, it is NOT linearly independent.)

One can continue the similarities with $\mathbb{R}^n$ by defining a subspace S of matrices to be a non-empty set of $m \times n$ vectors that satisfy properties (1) and (6) of Theorem 3.1.1. And we can also define a basis of such a subspace S to be a set $\mathcal{B}$ of $m \times n$ matrices that are both linearly independent and satisfying Span$\mathcal{B} = S$. But you'll undoubtedly be pleased to know that we won't be considering matrix subspaces and bases in this course. Instead, we will now start to look at the properties of a matrix that are not extensions of our knowledge of $\mathbb{R}^n$.